HARLAN J. BROTHERS AND JOHN A. KNOX

New Closed-Form Approximations to the Logarithmic Constant *e*

*Recently, the determination of n digits of π has become something of an industry [3; 10, pp. 62–63]. By contrast, however, few mathematicians seem interested in calculating the logarithmic constant e to comparable precision [7]. This area is underexplored, perhaps because in the case of e there is a straight-*forward Maclaurin series summation that is quite accurate.

In this article, we demonstrate that there exist alternative approximations to e that are also very accurate. We have found over 20 such approximations, all of which are elegant closed-form expressions obtainable using elementary calculus. We have used some of these approximations to calculate e to tens of thousands of decimal-place accuracy using commercially available software. Our most impressive result is a class of closed-form approximations with extremely rapid convergence that should outperform the familiar multiterm Maclaurin series approximation. Having been unable to find these approximations in a search of the published and electronic literature, we elaborate upon them here.

Traditional Approximations to *e*

The calculation of e has intrigued mathematicians for centuries. Joost Bürgi appears to have formulated the first approximation to e around 1620 [5, p. 31], obtaining three-decimal-place accuracy. Isaac Newton, in his *De Analysi* of 1669 [8, p. 235], published the first version of what is now known as the Maclaurin series expression for e^z, which for $z = 1$ is equal to

Direct:

$$\sum_{k=0}^{N} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{N!} \approx e. \quad (1)$$

Equation (1) is a "simple, direct approach [that] is the best way of calculating e to high accuracy" [2, p. 313]. Today, numerical values of e are derived using either optimized versions of this Maclaurin series [7, p. 157; S. Plouffe, personal communication] or the continued-fraction expansion approach pioneered by Euler [11, p. 1019].

An alternative approach to approximating e employs the Maclaurin series expression for $\ln(1 + x)$. This series was first discovered independently by Newton in about 1665 [6, p. 354] and Nicolaus Mercator in 1668 [7, pp. 38 and 74] and is valid on the interval $-1 < x \leq 1$:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \cdots. \quad (2)$$

Equation (2) can be used to obtain closed-form approximations to e that require the calculation of a single expression instead of a sum of $n + 1$ terms involving factorials as in Eq. (1). The only example of this alternative approach we have found in the literature sets $x = 1/x$ in Eq. (2) and multiplies the result by x to obtain

$$x \ln\left(1 + \frac{1}{x}\right) = 1 - \frac{1}{2x} + \frac{1}{3x^2} - \frac{1}{4x^3} + \frac{1}{5x^4} - \frac{1}{6x^5} + \frac{1}{7x^6} - \cdots. \quad (3)$$

Exponentiating and using the Maclaurin series for e^x leads to an approximation to e valid on the intervals $x < -1$ and $x \geq 1$, one that has been known by mathematicians and bankers alike since the early seventeenth century:

$$\text{Classical:} \quad \left(1 + \frac{1}{x}\right)^x = e\left[1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238\,043}{580\,608x^6} - \cdots\right]. \quad (4)$$

[In Eq. (4) and all similar equations later in this article, the right-hand side is the product of e and the bracketed quantities. The series expansion in Eq. (4) can be obtained in *Mathematica* using the following commands:

```
classical = Series[x Log[1 + 1/x] /.
                        x-> 1/y, y,0,6]

Collect [E^classical, E] /. y-> 1/x
```

Other series expansions in this article can be determined in like manner.]

To demonstrate how this approximation works, we insert $x = 100{,}000$ in Eq. (4) and obtain

$$\left(1 + \frac{1}{100{,}000}\right)^{100{,}000} \approx 2.71826\,82372. \quad (4')$$

From Eq. (4′), it is clear that for $x = 100{,}000$, the closed-form left-hand side of Eq. (4)—which we call the Classical method—yields an approximation to e that is accurate to four decimal places. In comparison to the Direct method, however, this is small potatoes; for example, Eq. (1) with $N = 16$ provides 14-decimal-place accuracy without too much more computational overhead. As a result, perhaps, closed-form approximations to e have received scant attention outside of the obligatory introductory-calculus discussion of Eq. (4) (e.g., [1], p. 558]).

Seven New Ways of Looking at e

Approximations to e far more accurate than the Classical method can be obtained via very similar methods. Below, we describe, in order of increasing accuracy, seven distinct algebraic expressions that approximate e for all $x > 1$. In each of Eqs. (4)–(14), it is the left-hand expression that is being proposed as an approximant to e.

1. *Complementary Classical Method (CCM):* The Classical method has a complementary form that results from letting $x = -x$ in Eq. (4):

$$\text{CCM:} \quad \left(1 - \frac{1}{x}\right)^{-x} = e\left[1 + \frac{1}{2x} + \frac{11}{24x^2} + \frac{7}{16x^3} + \frac{2447}{5760x^4} + \frac{959}{2304x^5} + \frac{238\,043}{580\,608x^6} + \cdots\right]. \quad (5)$$

CCM possesses virtually the same rate of convergence as the Classical method, but it approaches e from above, not below. Therefore, CCM can be combined with the Classical method to create new approximations to e that converge much more rapidly than either form by itself (see below).

2. *Complementary Addition Method (CAM):* A simple improvement results from adding the Classical method and CCM, and dividing the sum by 2:

$$\text{CAM:} \quad \frac{1}{2}\left[\left(1 + \frac{1}{x}\right)^x + \left(1 - \frac{1}{x}\right)^{-x}\right] = e\left[1 + \frac{11}{24x^2} + \frac{2447}{5760x^4} + \frac{238\,043}{580\,608x^6} + O\left(\frac{1}{x^8}\right) + \cdots\right]. \quad (6)$$

Note that this is, by the series analysis, the equivalent of the Classical method with all of the odd powers of x eliminated.

3. *Mirror-Image Method (MIM):* An approach similar to that used to derive CAM can also be used to create a distinct, and even more accurate, approximation to e. By replacing x with $2x$ in Eq. (3), adding this to Eq. (3) in which x is replaced with $-2x$, dividing by 2, and then exponentiating, we obtain

$$\text{MIM:} \quad \left(\frac{2x + 1}{2x - 1}\right)^x = e\left[1 + \frac{1}{12x^2} + \frac{23}{1440x^4} + \frac{1223}{362\,880x^6} + O\left(\frac{1}{x^8}\right) + \cdots\right]. \quad (7)$$

Like CAM, this eliminates all odd powers of x from the right-hand side (RHS), but MIM's coefficient for $1/x^2$ is smaller than in CAM. The derivation of MIM bears a striking resemblance to Gregory's series expansion for $\ln\,[(1 + x)/(1 - x)]$ [1, p. 661] and also to the series expansion for $\coth^{-1}x$ [2, p. 310]. To our knowledge, however, this approximation to e has never appeared in the literature.

4. *Power Ratio Method (PRM):* The Power Ratio Method was arrived at numerically by investigating the behavior of numbers that have been raised to their own power. Examination of the rate of change of the ratio between adjacent integer values of x that have been raised to the x power leads to the following approximation to e:

$$\text{PRM:} \quad \frac{(x + 1)^{x+1}}{x^x} - \frac{x^x}{(x - 1)^{x-1}} \equiv (x + 1)\left(1 + \frac{1}{x}\right)^x - (x - 1)\left(1 - \frac{1}{x}\right)^{-x} = e\left[1 + \frac{1}{24x^2} + \frac{11}{640x^4} + \frac{5525}{580\,608x^6} + O\left(\frac{1}{x^8}\right) + \cdots\right]. \quad (8)$$

As with CAM and MIM, PRM eliminates all odd powers of x and yields a rate of convergence of $O(1/x^2)$.

5. *CAM–MIM–PRM Amalgam Method (CMPAM):* Using the series expansions as a guide, a straightforward combination of CAM, MIM, and PRM can be created that achieves accuracy to $O(1/x^6)$, five orders of magnitude better than the Classical method:

$$\text{CMPAM:}\quad \frac{1}{1209}(1941\ PRM - 679\ MIM - 53\ CAM) = e\left[1 - \frac{383\,443}{83\,566\,080x^6} + O\left(\frac{1}{x^8}\right) + \cdots\right]. \quad (9)$$

A variety of other forms with better-than-Classical accuracy may be formed in this manner, but usually at the cost of algebraic elegance and computational time; we examine them elsewhere. Below, we explore more adroit ways to increase the accuracy of these approximations.

6. *Brothers–Knox Method (BK):* Drawing on the ideas inherent in MIM and PRM, an extremely rapidly convergent class of approximations to e can be created by substituting a^x for x in MIM and then dividing the numerator and denominator by a^x:

$$\text{BK:}\quad \left(\frac{2 + a^{-x}}{2 - a^{-x}}\right)^{a^x} = e\left[1 + \frac{1}{12a^{2x}} + \frac{23}{1440a^{4x}} + \frac{1223}{362\,880a^{6x}} + O\left(\frac{1}{a^{8x}}\right) + \cdots\right]. \quad (10)$$

One special case of BK seems especially well suited to computational analysis:

$$\left(\frac{1 + 2^{-x}}{1 - 2^{-x}}\right)^{2^{x-1}} = e\left[1 + \frac{1}{3(2^{2x})} + \frac{23}{90(2^{4x})} + \frac{1223}{5670(2^{6x})} + O\left(\frac{1}{2^{8x}}\right) + \cdots\right]. \quad (11)$$

Another special case of BK, in which $a = x$ and x is replaced by $2x$, provides exceptionally rapid convergence to e:

$$\left(\frac{2x^x + x^{-x}}{2x^x - x^{-x}}\right)^{x^{2x}} = e\left[1 + \frac{1}{12(x^{4x})} + \frac{23}{1440(x^{8x})} + \frac{1223}{362\,880(x^{12x})} + O\left(\frac{1}{x^{16x}}\right) + \cdots\right]. \quad (12)$$

As $x^{4x} >> x!$ for integer values of $x > 1$, the RHS of Eq. (12) implies that BK should converge to e much more quickly than the Direct method when the two methods are compared using $N = x$ in Eq. (1). Furthermore, the comparative advantage of BK versus the Direct method will only widen for increasing x.

7. *Hyperexponentiated Brothers–Knox Method (BK^n):* Obviously, the BK method can be generalized for $a = x$; x replaced by $2x$, and an *arbitrary* number of exponentiations:

$$\text{BK}^n:\quad \left(\frac{2x^{x^n} + x^{-(x^n)}}{2x^{x^n} - x^{-(x^n)}}\right)^{x^{(2x^n)}} = e\left[1 + \frac{1}{12(x^{4x^n})} + \cdots\right], \quad (13)$$

in which n indicates the order of the exponentiation. BK^n is, in fact, a generalization of other approximations presented here; for example, BK in the form of Eq. (12) corresponds to BK^n with $n = 1$; MIM corresponds to the derivation of BK^n with $a = x^{1/2}$, and $n = 0$.

The most rapidly converging example of this class of approximations results when n is set equal to x in BK^n (or equivalently, when $a = x$ and x is replaced by $2x^x$ in BK):

$$\text{BK}^x:\quad \left(\frac{2x^{x^x} + x^{-(x^x)}}{2x^{x^x} - x^{-(x^x)}}\right)^{x^{(2x^x)}} = e\left[1 + \frac{1}{12(x^{4x^x})} + \cdots\right], \quad (14)$$

The convergence of BK^x is astonishingly rapid; it would appear to possess greater-than-quadratic convergence even for small x [4, pp. 70–71].

The ultimate example of hyperexponentiation would employ in BK the relations $a = x$ and $x = x \uparrow\uparrow x$, where our notation follows Knuth's [9, p. 38] and indicates that x is to be raised to the x power x number of times. We do not pursue this here. Although this approach of hyperexponentiation closely resembles work on infinite iterated exponentials [http://www.mathsoft.com/asolve/constant/itrexp/itrexp.html], we are not aware of any application of the latter to the calculation of e.

Numerical Computations

For the visually inclined, we provide two figures and a table which illustrate the utility of our new closed-form approximations to e.

Figure 1 compares the Classical and Direct methods to

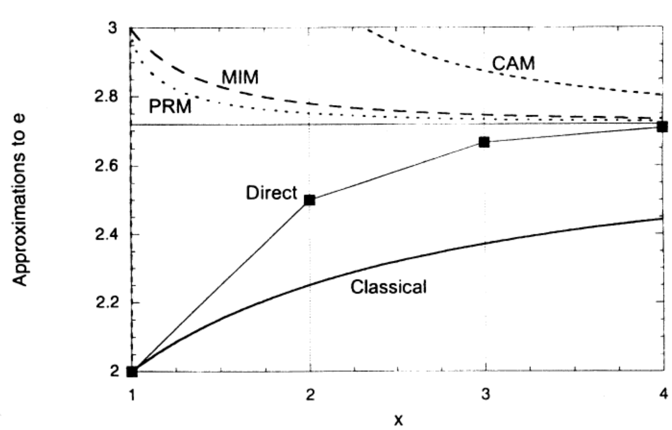

Figure 1. A comparison of the new approximations CAM, MIM, and PRM versus the Classical and Direct methods for $1 \le x \le 4$. All methods are defined in the text. The Direct method is calculated using Eq. (1) in which $N = x$ and, therefore, is the sum of $x + 1$ terms; all other methods are closed-form approximations. A 15-decimal-place-accurate approximation to e is plotted for visual reference.

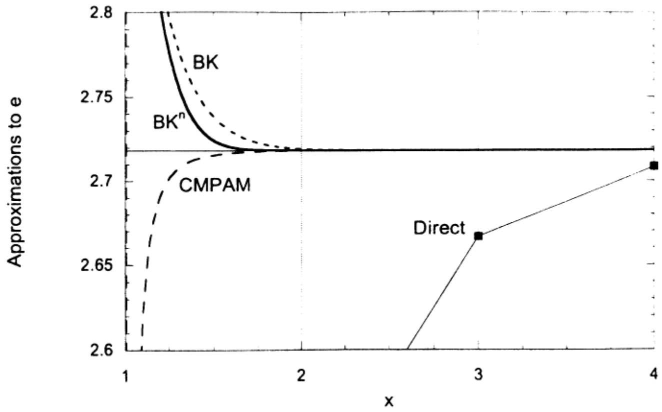

Figure 2. A comparison of the new approximations CMPAM, BK, and BK^n, where $n = 2$, versus the Direct method for $1 \le x \le 4$. A 15-decimal-place-accurate approximation to e is plotted for visual reference. Note that the ordinate scale is not the same as in Figure 1.

CAM, MIM, and PRM for small x. The direct method (1) is computed where $N = x$. It is noteworthy that all of the new approximations shown here tend to e from above, whereas the Direct and Classical methods approach e from below. The superior convergence of CAM, MIM, and PRM versus the Classical method is obvious; however, for $x \geq 5$, the Direct method is more accurate than these three new approximations.

In Figure 2, the Direct method is compared to CMPAM, BK, and BK^n for small x. The Direct method is calculated as in Figure 1; BK is calculated using the special case (12); and BK^n is Eq. (13) with $n = 2$. The extremely rapid convergence of CMPAM, BK, and BK^n clearly outpaces that of the Direct method for small x.

Table 1 presents a comparison of all the approximation methods for larger values of x than those shown in the figures. In Table 1, BK and BK^n are calculated as in Figure 2.

In addition, we have made a very modest foray into the realm of high-precision calculations of e. We have used MIM for this purpose, as it involves very few arithmetic operations and performs well without optimization. Running *Mathematica 2.2* on an IBM RS/6000 computer, we calculated MIM in the form $[(x + 1)/(x - 1)]^{x/2}$. This form of MIM is chosen to mitigate the loss of precision that occurs if the calculation is done with MIM as shown in Eq. (7). For the same reason, we manually typed in x as $10^{15,000}$ with 30,000 decimal places; in *Mathematica*, there is loss of precision if one defines x as `x → N[10,30000]^15000` instead. (Manual typing of 30,000 decimal places is not an onerous task using *Mathematica*'s paste option.)

Once x was defined, we employed MIM to calculate 29,999 correct decimal places of e in 15 s. This compares very favorably to the 12-s runtime on the same hardware and software needed to calculate an equivalent number of correct decimal places using the `N[Exp[1],30000]` command in *Mathematica*. This is by no means a rigorous test of the computational speed and accuracy obtainable with our new approximations using this software/hardware configuration, but is presented to give some comparison between existing and new methods.

Given the extraordinarily rapid convergence of BK^n, we believe that it may be possible to use our methods to compute e to unprecedented accuracy. As an extreme example, we estimate that for $x = 10$, the BK^x approximation would yield e accurate to 40 billion decimal places, although obviously that computational task would be formidable. However, the computational potential of the BK^n class of approximations cannot be thoroughly evaluated until optimization of the calculation of $x^{(x^n)}$ using Fast Fourier Transform (FFT) methods is performed [S. Plouffe, personal communication]. We leave these experiments to the experts on this subject.

Discussion

We have identified and formally established the existence of new closed-form approximations to e. Six of the new approximations discussed here improve upon the classical closed-form approximation. In particular, the BK^n class of approximations converges to e much more rapidly than even the direct Maclaurin series method. Therefore, our work may have practical application.

The impressive numerical accuracy of these new approximations should not cloud our eyes to an even more extraordinary aspect: the elegance and simplicity of the expressions for e, particularly MIM. Compared to many other methods for computing classical constants, MIM is breathtaking. Only one addition, one subtraction, one multiplica-

TABLE 1. Comparison of decimal-place accuracy of various approximations to e

x (=N)	Classical	CAM	MIM	PRM	CMPAM	Direct	BK	BK^n
10	0	1	2	2	7	7	~40	400
100	1	3	4	4	13	159	~800	~80,000
1000	2	5	6	6	19	2570	~12,000	~12,000,000

Note: To illustrate this comparison, in the first row where $x = N = 10$, the Classical method = $(1 + 1/10)^{10}$, MIM = $(21/19)^{10}$, and the Direct method = $\sum_{k=0}^{10} (1/k!)$. BK^n is calculated with $n = 2$. The ~ sign indicates a theoretical estimate.

tion (employed twice), one division, and one exponentiation are required to approximate e to tens of thousands of decimal places. The mathematical knowledge required to understand it is provided in introductory calculus, but the end result can be grasped and computed by an elementary-school student. The logarithmic constant e is famous for turning up whenever natural beauty and mathematical elegance commingle. Our work provides a new glimpse of its austere charm.

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