Section 1.1: Statements, Symbolic Representations, and Tautologies

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Abstract

We encounter the elements of logic: statements, connectives, tautologies, contradictions, etc., and create well-formed formulas (wffs - "whiffs") from these basic elements. An algorithm for detecting tautologies in the form of implications is described.

Note: dual labelled exercises refer to 5th/6th edition numbers. Hence #26/29 refers to problem 26 in the 5th edition, and 29 in the 6th edition.

- **Statement/proposition**: a sentence possessing truth value (T or F).

  Exercise #1

  a. The moon is made of green cheese. ✓ F
  b. He is certainly a tall man. ✗ Don't know what "he" is.
  c. Two is a prime number. ✓ T
  d. Will the game be over soon? ✗ It's a question!
  e. Next year interest rates will rise.
  f. Next year interest rates will fall.
  g. $x^2 - 4 = 0$ ✗ Don't know what "x" is (x is a variable!)

  A couple of things we should observe in examining these examples:

  – "Truth is relative" (requires a **context**).
  – Variables come in several flavors.
  – English is a troublesome language!

- **Logical connectives** join statements into **formulas**, or arguments, or compound statements:

  – conjunction (symbolized by $\wedge$, "and")

– disjunction (symbolized by $\vee$, "or")

– implication (symbolized by $\longrightarrow$)

– equivalence (symbolized by $\longleftrightarrow$, "if and only if")

– negation (symbolized by ' – "not" – which is a *unary* operation)

Note: These connectives are not independent - some of these may be derived from the others (Exercise #29/33 shows that conjunction and negation suffice to write the others, for example).

A	B	$A \wedge B$	$A \vee B$	$A \longrightarrow B$	A'	B'	$A \leftrightarrow B$	$B \to A$
T	T	T	T	T	F	F	T	T
T	F	F	T	F	F	T	F	T
F	T	F	T	T	T	F	F	F
F	F	F	F	T	T	T	T	T

$A \leftrightarrow B \iff (A \to B) \wedge (B \to A)$

Does the table for implication seem weird to you? It's by convention!

In the implication $A \longrightarrow B$, A is the **antecedent**, and B is the **consequent**. Some English equivalents to implication are:

– If A, then B.

– A implies B.

– A, therefore B.

– A only if B.

– B follows from A.

– A is a sufficient condition for B.

– B is a necessary condition for A.

Implication plays an especially important role among connectives, so learn it well!

Exercise #4

a. Healthy plant growth follows from sufficient water.

b. Increased availability of information is a necessary condition for further technological advances.

c. Errors will be introduced only if there is a modification of the program.

d. Fuel savings implies good insulation or storm windows throughout.

Exercise #6/7ade: Negating implications

$(A \to B)'$

A : the price is high
B : (food is good) $\wedge$ (service is excellent)

If the price is high then the food is good and the service is excellent.

(price is high) $\wedge$ [(food is good) $\wedge$ (service is excellent)]$'$

$A \wedge B'$

- **Well-formed formula** (wff - "whiff") is a compound statement made up of statements, logical connectives, and other wffs

  What makes one well-formed? There are just a few rules for creating wffs:

  a. Statements A, B, etc., are wffs, as are the following:
  b. $(A \vee B)$,
  c. $(A \wedge B)$,
  d. A',
  e. $(A \longrightarrow B)$,
  f. $(A \Longleftrightarrow B)$,

  A & B themselves can be wffs in these rules

  – **Order of precedence**:
    * parentheses
    * $'$
    * conjunction, disjunction
    * implication
    * equivalence

  Order of precedence helps us to simplify our lives: hence,

$$A \wedge B \longrightarrow C \text{ means } (A \wedge B) \longrightarrow C$$

  – **main connective** (last to be applied)

- **Truth table** for a wff with n statement letters: 2^n rows

  Example: the table for implication above, which is a binary (2 statement letter) logical connective. Hence there are $2^2 = 4$ rows.

- **tautology**: wff which is always true (represented by 1).

- **contradiction**: wff which is always false (represented by 0).

- **equivalent wffs**: wffs A and B are equivalent, $A \iff B$, if the wff

  $$A \longleftrightarrow B$$

  is a tautology. (*How can we prove that?*)

  Some tautological equivalences:

1a. $A \vee B \iff B \vee A$	1b. $A \wedge B \iff B \wedge A$	Commutative
2a. $(A \vee B) \vee C \iff A \vee (B \vee C)$	2b. $(A \wedge B) \wedge C \iff A \wedge (B \wedge C)$	Associative
3a. $A \vee (B \wedge C) \iff (A \vee B) \wedge (A \vee C)$	3b. $A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$	Distributive
4a. $A \vee 0 \iff A$	4b. $A \wedge 1 \iff A$	Identity
5a. $A \vee A' \iff 1$	5b. $A \wedge A' \iff 0$	Complement

  Equivalent wffs will be useful when we are proving arguments, and want to replace complex wffs with simpler ones.

- **De Morgan's Laws** are two specific examples of equivalent wffs:

  - $(A \vee B)' \iff A' \wedge B'$
  - $(A \wedge B)' \iff A' \vee B'$

  Hence we claim that $(A \vee B)' \longleftrightarrow (A' \wedge B')$ is a tautology.

  Notice that the two formulas of De Morgan's Laws appear analogous ("dual"). In fact, one is the negation of the other.

  Question: How so? $(A' \wedge B')' \iff (A')' \vee (B')'$

  $\iff A \vee B$ (double negation)

  negate both sides:

  $A' \wedge B' \iff (A \vee B)'$

Table 1: Exercise #17/20e: Verify by constructing a truth table that the following is a tautology: $(A \vee B)' \longleftrightarrow A' \wedge B'$.

A	B	$A \vee B$	$(A \vee B)'$	A'	B'	$A' \wedge B')$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Exercise #24/27

V : Value1 < Value2

N : odd(Number)

If $[(V \vee N)' \vee (V' \wedge N)]$ then ....

$(V' \wedge N') \vee (V' \wedge N)$

$V' \wedge (N' \vee N)$

$V' \wedge 1$

V'

If V' then

....

- **Algorithm**: a set of instructions that can be mechanically executed in a finite amount of time in order to solve some problem.

  Often written out in **pseudocode**, the author provides us an example of an algorithm: TautologyTest, which is useful for whether or not an implication (that is, a wff where the main connective is implication) is, in fact, always true (a tautology). She proceeds by contradiction (one proof technique we'll study further in Chapter 2): assume that the implication $P \rightarrow Q$ is false. Then P must be true, and Q false (the only scenario which makes an implication false).

Exercise 26/29: c

Building a truth table for the implication also constitutes an algorithm to test to see if it is true, but, although the truth table algorithm may be more powerful (as more general, working for all would-be tautologies), an algorithm like TautologyTest may be faster when applied to a particular implication.