

Section 2.2: Induction

February 5, 2009

Abstract

In this section we investigate a powerful form of proof called **induction**. This is useful for demonstrating that a property, call it $P(n)$, holds for all integers n greater than or equal to 1.

Actually, the “1” above is not essential: any “base integer” will do (like 0, for example: it really only matters that there be a “ground floor”, or “anchor”).

1 Induction

Induction is a very beautiful and somewhat subtle method of proof: the idea is that we want to demonstrate a property associated with natural numbers (or a subset of the natural numbers). As a typical example, consider a theorem of the following type (which we might call “Gauss’s theorem,” hypothesized when he was seven or so):

Prove that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

An induction proof goes something like this:

- We’ll show that it’s true for the first case (usually $k = 1$, called the base case). While the first case is often $k = 1$, this isn’t mandatory: we simply need to be sure that there is a first case for which the property is true. $k = 0$ is another popular choice....
- Then we’ll show that, if the property is true for the k^{th} case, then it’s true for the $(k + 1)^{\text{th}}$ case (the inductive step).
- Then we’ll put them together: if it’s true for 1, then it’s true for 2; if it’s true for 2, then it’s true for 3; “to infinity, and beyond!” Or up the ladder, as our author would say.

Imagine dominoes falling. That’s what it’s like.

The most commonly used form of the principle of induction is expressed as follows:

First Principle of Mathematical Induction:

1. $P(1)$ is true
 2. $(\forall k)[P(k) \text{ true} \rightarrow P(k+1) \text{ true}]$
- $$\left. \begin{array}{l} 1. P(1) \text{ is true} \\ 2. (\forall k)[P(k) \text{ true} \rightarrow P(k+1) \text{ true}] \end{array} \right\} \rightarrow P(n) \text{ true for all positive integers } n$$

or, more succinctly,

$$P(1) \wedge (\forall k)[P(k) \rightarrow P(k+1)] \rightarrow (\forall n)P(n)$$

where the domain of the interpretation is the natural numbers. This is just *modus ponens* applied over and over again. Put *modus ponens* into an infinite loop, because we want it to run off to infinity! This might be the first infinite loop you've ever liked!

Vocabulary:

- **inductive hypothesis:** $P(k)$
- **basis step** (base case, anchor): establish $P(1)$
- **inductive step** (implication): $P(k) \rightarrow P(k+1)$

Example: (Practice 7, or "Gauss's theorem") Prove that, for any natural number n , $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.

$$P(n) : 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

① Anchor case : $P(1) : 1 = \frac{1(1+1)}{2}$ ✓

② Implication : $P(k) \rightarrow P(k+1)$

Assume $P(k) : 1 + \dots + k = \frac{k(k+1)}{2}$
Need to: Establish $P(k+1) : 1 + \dots + k + k + 1 = \frac{(k+1)(k+1+1)}{2}$

Consider $1 + \dots + k + k + 1 = \frac{k(k+1)}{2} + k + 1$ (by $P(k)$)

$\therefore P(n) \forall n \in \mathbb{N} = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+1+1)}{2}$ ✓

Example: Exercise 34, p. 114: Prove that $2^{n-1} \leq n!$ for $n \geq 1$.

① Anchor : $P(1) : \underbrace{2^{1-1}}_{2^0} \leq 1! \quad ?$
 $2^0 \leq 1 \quad 1 \leq 1$ ✓

② \Rightarrow : Assume $P(k) : 2^{k-1} \leq k!$, + show
 $P(k+1) : 2^{(k+1)-1} \leq (k+1)!$

Consider $2^{(k+1)-1} = 2 \cdot 2^{k-1} \leq 2(k)! \leq (k+1)k! \leq (k+1)k!$ for $k \geq 1$
 $\leq (k+1)!$ ✓

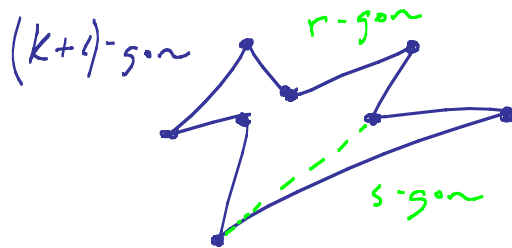
QED : the result is true for all $n \geq 1$.

A second (and seemingly more powerful) form of induction is given by the **Second Principle of Mathematical Induction**:

1. $P(1)$ is true
 2. $(\forall k)[P(r)$ true for all $r, 1 \leq r \leq k \rightarrow P(k+1)$ true]
- $$\left. \begin{array}{l} 1. P(1) \text{ is true} \\ 2. (\forall k)[P(r) \text{ true for all } r, \\ 1 \leq r \leq k \rightarrow P(k+1) \text{ true}] \end{array} \right\} \rightarrow P(n) \text{ true for all positive integers } n$$

This principle is useful when we cannot deduce $P(k+1)$ from $P(k)$ (for k alone), but we can deduce $P(k+1)$ from all preceding integers, beginning at the base case.

Example: Exercise 66b, p. 116.



① Anchor ($n=3$):
 triangle; sum of interior angles is 180° ,
 $+ (3-2) \cdot 180^\circ = 180^\circ \checkmark$

② $P(k)$: sum of interior angles is $(k-2) \cdot 180^\circ$. We assume $P(r)$ for $3 \leq r \leq k$, + must show $P(k+1)$: sum is $(k+1-2) \cdot 180^\circ$.

$$r + s = (k+1) + 2$$

Sum of angles of $(k+1)$ -gon is the sum of
 " " " " r -gon +
 " " " " s -gon

$$\begin{aligned} (r-2)180 + (s-2) \cdot 180 &= (r+s-4) \cdot 180 \\ &= ((k+1)-2) \cdot 180^\circ \checkmark \end{aligned}$$

QED: the result holds for $n \geq 3$.

In spite of appearances, these two principles are equivalent; furthermore they are also equivalent to the **Principle of Well-Ordering**, which states that every collection of positive integers that contains any members at all has a smallest member.

Example: Prove that the first principle of induction implies well-ordering.

A Couple of Fun Examples:

(a) The prisoner's last request (finite backwards induction!)

(b) Now that we understand induction, let's use it to prove an amazing fact: All horses are the same color.

Proof: By induction, on the number of horses.

Base case: 1 horse. No problem! Same color.

Inductive step: we'll show that if it is true for any group of N horses, that all have the same color, then it is true for any group of $N + 1$ horses.

Well, given any set of $N+1$ horses, if you exclude the last horse, you get a set of N horses. By the inductive step these N horses all have the same color. But by excluding the first horse in the pack of $N+1$ horses, you can conclude that the last N horses also have the same color. Therefore all $N+1$ horses have the same color.

QED - or have we?

$$\#28. \quad n^2 > n+1 \quad \text{for } n \geq 2$$

$$\textcircled{1} \text{ Anchor : } n = 2$$

$$2^2 = 4 ; \quad 2+1 = 3 ; \quad 4 > 3 \quad \checkmark$$

$$2^2 \stackrel{?}{>} 2+1 ; \quad \text{Yes } 4 > 3 \quad \checkmark$$

$$\textcircled{2} \Rightarrow : \quad \text{Assume } P(k) : k^2 > k+1 \\ + \text{ Show } P(k+1) : (k+1)^2 > (k+1)+1$$

$$\text{Consider } (k+1)^2 = k^2 + 2k + 1$$

$$> k+1 + 2k + 1$$

$$> (k+1) + 1 \quad (k \geq 2)$$

✓

QED : The result holds by the 1st Principle of Induction for all $n \geq 2$.