Section 4.1: Relations

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Abstract

This section, the only section we consider from Chapter 4, simply gives us some basic vocabulary and notions of relations (especially equivalence relations) that we will need when we get to Boolean algebras later.

1 Binary Relations

We begin by remembering the notion of a **binary operation** from the previous chapter:
Definition: $\circ$ is a **binary operation** on a set S if for every ordered pair (x, y) of elements of S, $x \circ y$ exists, is unique, and is a member of S.

Our first new definition is a related notion:
Definition: A **binary relation** ρ on a set S is a subset of the Cartesian product $S \times S$. We say that x is related to y by ρ, and

$$x \; \rho \; y \iff (x, y) \in \rho$$

Example: Practice 1, p. 288.

a. $x \rho y \Leftrightarrow x = y+1$; $(3,2) \in \rho$

b. $x \rho y \leftrightarrow x$ divides y : $(2,4), (2,6) \in \rho$

c. $x \rho y \leftrightarrow x$ odd ; $(3,4), (5,6) \in \rho$

d. $x \rho y \leftrightarrow x > y^2$; $(2,1), (5,2) \in \rho$

Binary relations come in various flavors (see the figures on p. 288):

- one-to-one ✓ a function
- one-~~to~~-many fails vertical line test
- many-to-one ✓ a function (e.g. quadratic)
- many-to-many failing VLT

Which of the forms above correspond to the functions we know from calculus class? Functions are relations, but with certain constraints.

Properties of Relations: Let ρ be a binary relation on a set S:

- ρ is **reflexive** $\iff$ $(\forall x)\,(x \in S \to (x, x) \in \rho)$
- ρ is **symmetric** $\iff$ $(\forall x)(\forall y)\,(x \in S \wedge y \in S \wedge (x, y) \in \rho \to (\cancel{x}\,y, \cancel{y}\,x) \in \rho)$
- ρ is **transitive** $\iff$ $(\forall x)(\forall y)(\forall z)\,(x \in S \wedge y \in S \wedge z \in S \wedge (x, y) \in \rho \wedge (y, z) \in \rho \to (x, z) \in \rho)$
- ρ is **antisymmetric** $\iff$ $(\forall x)(\forall y)\,(x \in S \wedge y \in S \wedge (x, y) \in \rho \wedge (y, x) \in \rho \to x = y)$

Example: Practice ~~8, p. 288~~. 5, p 291.

a. $S = \mathbb{N}$; $x + y$ even — R T S

b. $S = \mathbb{Z}^+$: x divides y — R T A

c. $S =$ all lines in plane : $x \parallel y$ — R T S

d. $S = \mathbb{N}$: $x = y^2$ — ~~R~~

e. $S = \{0, 1\}$: $x = y^2$ — R S T A

f. $S =$ people in Peoria : x older than y — ~~R~~

g. $S =$ students in class : x in same row as y

h. $S = \{1, 2, 3\}$: $\rho = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

2 Equivalence Relations

Definition: An **equivalence relation** on a set S is a reflexive, symmetric, and transitive binary relation.

Examples:

- Parallel lines
- Affine functions

$x \rho y \leftrightarrow x \parallel y$

$$\left.\begin{aligned} f(x) &= ax + b \\ g(x) &= ax + c \end{aligned}\right\} \text{ both have graphs that are parallel lines}$$

$f \rho g$

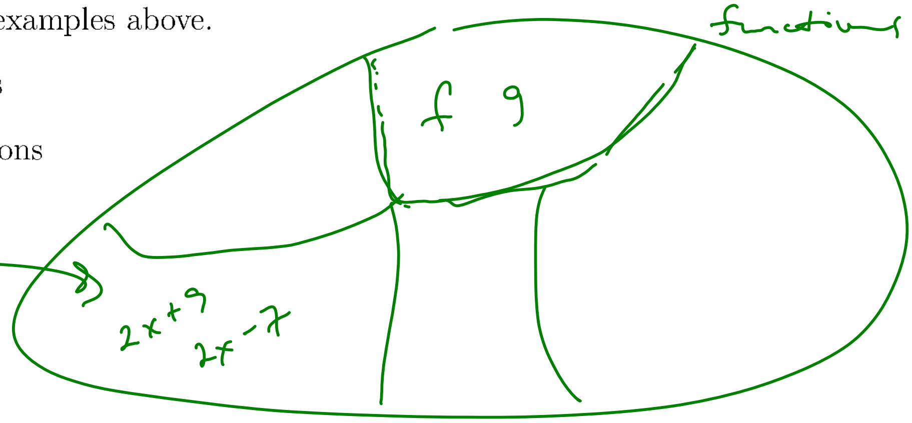
Definition: A **Partition** of a set S is a collection of nonempty disjoint subsets of S whose union equals S.

Theorem: An equivalence relation ρ on a set S determines a partition of S, and a partition of a set S determines an equivalence relation on S.

Example: Same examples above.

- Parallel lines
- Affine functions

all affine functions



An equivalence class