Section 5.1: Graphs and Their Representations

March 6, 2009

Abstract

We are introduced to definitions of graphs, various kinds of graphs, characteristic features of graphs, and even a few theorems about graphs (for example, we learn when two graphs are the same, or isomorphic, even when they look strikingly different).

We then take a look at planar graphs (in particular at Euler's formula), and computer representations of graphs (adjacency matrices, adjacency lists).

1 Definitions

A graph is defined loosely as a set of nodes, and a set of arcs which connect some of the nodes.

More formally, we have the following

Definition: a **graph** is an ordered triple (N, A, g) where

$N =$	a nonempty set of nodes, or vertices
$A =$	a set of arcs, or edges
$g =$	a function associating each arc a with an *unordered* pair {x,y} of nodes.

x and y are the endpoints of the arc. g is a function $g : A \to \{\{x, y\} | x \in N \text{ and } y \in N\}$.

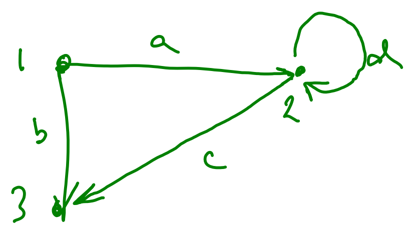
Example: Practice #1, p. 404.

Definition: a **directed graph** is an ordered triple (N, A, g) where

$N =$	a nonempty set of nodes, or vertices
$A =$	a set of arcs, or edges
$g =$	a function associating each arc a with an *ordered* pair (x,y) of nodes.

so g is a function $g : A \to \{(x, y) | x \in N \text{ and } y \in N\}$.

Example: Exercise #1, p. 423.



$g(a) = (1,2)$ $\qquad$ $g(d) = (2,2)$

$g(b) = (1,3)$

$g(c) = (2,3)$

2 Examples of graphs in action (p. 405)

- Road map of Arizona
- Ozone Molecule
- "data flow diagram" for state auto licensing office
- "star topology" for network
- neural network
- Map of Rabies-infected towns in Connecticut

3 Graph Terminology

Take a moment to draw a graph – that is an object consistent with the above definition(s). Using the graph terminology handout to classify your graph. In particular the vocabulary we want to focus on is as follows:

- degree of a vertex
- adjacent vertices

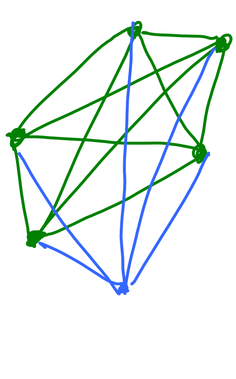
- parallel edges
- loops
- simple
- complete
- path
- cycle
- reachable
- connected

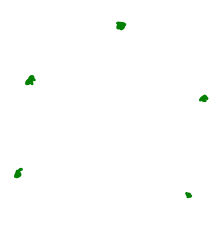
Example: Exercise #2, p. 423.

4 Special Graphs

By K_n we will understand the simple, complete graph with n nodes.

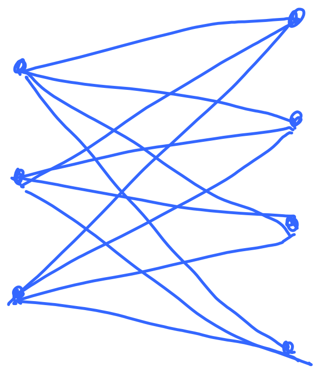
Example: Exercise #5a, p. 423 Draw K_6.





A **bipartite complete graph** $K_{m,n}$ is a graph of N nodes which break into two groups, N_1 and N_2, of size m and n respectively, with the property that two nodes x and y are adjacent $\iff$ $x \in N_1$ and $y \in N_2$.

Example: Exercise #5b, p. 423: Draw $K_{3,4}$.



5 Isomorphic Graphs

The idea of isomorphism is that two structures can be "morphed" into each other (they are in some sense identical, up to labelling). Our objective, in general, is to figure out the "morphism" (isomorphism - same form!).

Example: Look at Figure 5.17, p. 411: can you morph the two graphs together?

Definition: Two graphs (N_1, A_1, g_1) and (N_2, A_2, g_2) are **isomorphic**

if there are bijections (one-to-one and onto mappings) $f_1 : N_1 \to N_2$ and $f_2 : A_1 \to A_2$ such that for each arc $a \in A_1$,

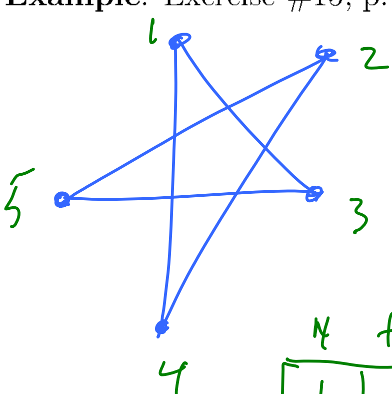
$$g_1(a) = \{x, y\} \iff g_2[f_2(a)] = \{f_1(x), f_1(y)\}$$

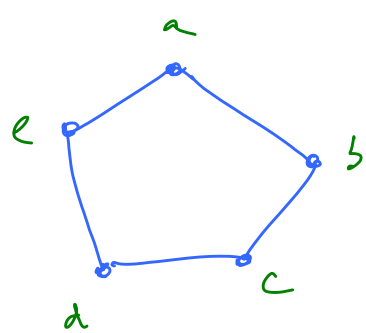
(replace braces by parentheses for a directed graph).

Example: Practice #7, p. 412. If you managed to morph the two graphs in Figure 5.17, then you should be able to "see" the rest of function f_2.

Theorem: Two simple graphs (N_1, A_1, g_1) and (N_2, A_2, g_2) are isomorphic if there is a bijection $f : N_1 \to N_2$ such that for any nodes n_i and n_j of N_1, n_i and n_j are adjacent $\iff$ $f(n_i)$ and $f(n_j)$ are adjacent.

Example: Exercise #15, p. 425.





N	f(N)
1	a
2	c
3	e
4	b
5	d

Here are some tests for determining when two graphs are **not** isomorphic:

(1) The graphs don't have the same number of nodes.

(II) The graphs don't have the same number of arcs.

(III) One graph is connected and the other isn't.

(IV) One graph has a node of degree k and the other doesn't.

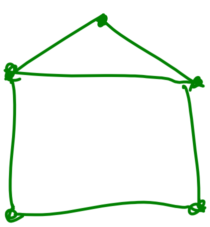
(V) One graph has parallel arcs and the other doesn't.

(VI) One graph has loops and the other doesn't.

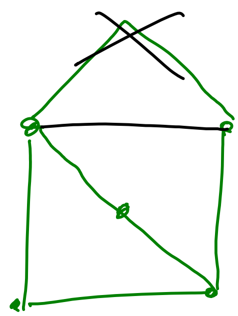
(VII) One graph has cycles and the other doesn't.

This list is not complete, however: sometimes things get trickier than this (as shown in Example 12, p. 413).

Example: Exercise #12, p. 425.



(a)



(d)

1) no cycle of length 3

2) the two degree three nodes are not connected by an arc.

6 Planar Graphs

A **planar graph** is one which can be drawn in two-dimensions so that its arcs intersect only in nodes. "Designers of integrated circuits want all components in one layer of a chip to form a planar graph so that no connections cross." (p. 413)

Example: Revisit #15, p. 425.

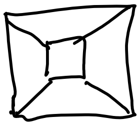
$r = 2$

$a = 5$

$n = 5$

$r - a + n = 2$

planar graph.



$\boxed{6 - 12 + 8 = 2}$

Euler's Formula for simple, connected planar graphs states that

$$r - a + n = 2$$

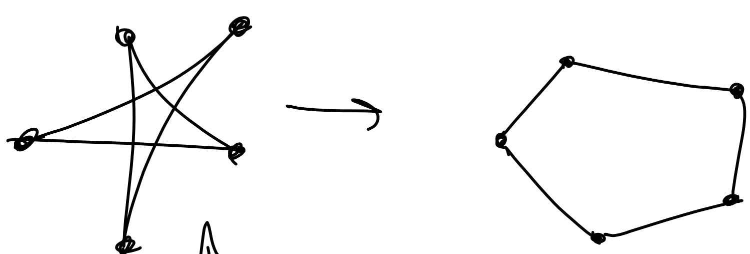
where n is the number of nodes, a is the number of arcs, and r is the number of regions (including the infinite region surrounding the graph).

Think "ran to" to remember the formula....

Check out the author's proof of the theorem (p. 414): Hey! What's induction doing in here? Euler's formula is proven by induction, on a, the number of arcs, and a consideration of cases (node of degree 1; no node of degree 1).

Note: about Euler (Born: 15 April 1707 in Basel, Switzerland Died: 18 Sept 1783 in St Petersburg, Russia). He was so prolific that his work is still being compiled. He went blind in his old age, and became even more prolific! He was an incredible calculating machine.

Example: Revisit #15, p. 425., for a check.



Two regions inside + outside

$\underline{\underline{2 - 5 + 5 = 2}}$

No ~~arbitrary~~ intersections
(extraneous)

The following theorem provides some estimates on the relationship between the number of arcs and nodes that a planar graph may possess:

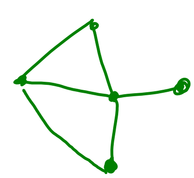
Theorem: For a simple, connected, planar graph with n nodes and a arcs,

(I) If the planar representation divides the plane into r regions, then

$$n - a + r = 2$$

(II) If $n \geq 3$, then

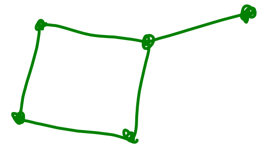
$$a \leq 3(n - 2)$$



$$2a \geq 3r = 3(2 - n + a) = 6 - 3n + 3a$$

$3n - 6 \geq a \implies a \leq 3(n-2)$

(iii) If $n \geq 3$ and there are no cycles of length 3, then



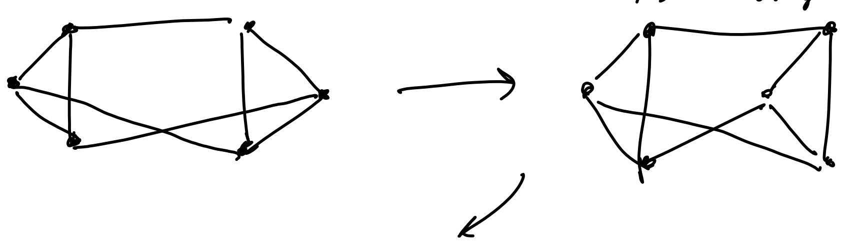
$$a \leq 2(n-2)$$

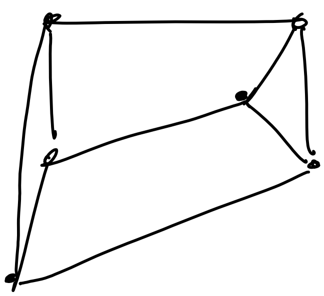
From this theorem we can deduce that K_5 is not planar, since it has 5 nodes, and 10 arcs, and $10 > 3 \cdot 3$.

Also from this theorem we can deduce that $K_{3,3}$ is not planar, since it has 6 nodes, and 9 arcs, and no cycles of length 3: $9 > 2 \cdot 4$.

Example: Exercise #26, p. 427.

a sequence of isomorphic graphs.





$r - a + n = 2$

$5 - 9 + 6 = 2$

7 Computer Representations of Graphs

We want to examine two different representations of graphs by a computer:

- the adjacency matrix, and
- the adjacency list.

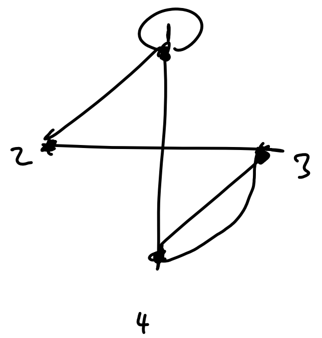
A **matrix** is basically a spreadsheet: a rectangular data set of numbers indexed by rows and columns.

An **adjacency matrix** for a graph with N nodes is of size N by N, where the rows and columns of the matrix represent the vertices. If the graph is undirected, then the element a_{ij} of the matrix is non-zero $\iff$ nodes i and j are adjacent; if directed, then the element a_{ij} of the matrix is non-zero $\iff$ there is an arc **from** node i **to** node j.

In our textbook, the element of the matrix $a_{ij} = p$, the number of arcs meeting the criteria above.

Example: Practice #16, p. 419.

symmetric b/c undirected



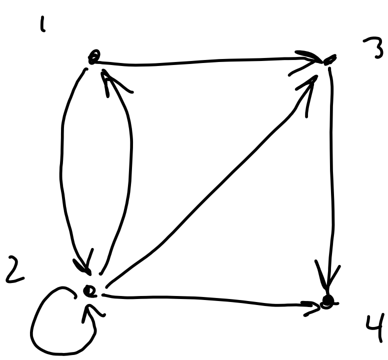
$$A = \begin{array}{cc} & \text{to} \\ \text{from} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

loops on the diagonal

For an undirected graph the adjacency matrix is **symmetric** (which means that we can reduce storage by about half); for a directed graph, the matrix may well be unsymmetric.

Let's look at a nice web page, with an example of a directed graph

Example: Exercise #~~37~~ 40, p. 428.

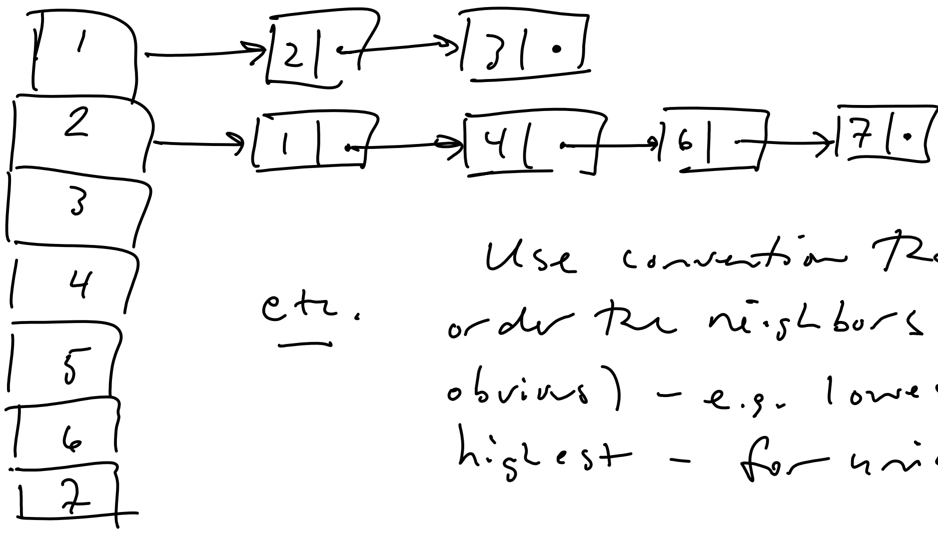


$$A = \begin{array}{cc} & \text{to} \\ \text{from} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

- This 1990 commuting patterns page might be modelled as a directed, weighted graph. Its adjacency matrix would be exactly the numerical portion of this table, and it would be a *full* matrix.
- A map of Rabies-infected towns in Connecticut gives rise to an undirected graph. The towns are nodes, and an arc is created if two towns are adjacent. This will lead to a *sparse* symmetric adjacency matrix, however, as very few towns are adjacent to any particular town.

An **adjacency list** might be a better storage method for graphs with relatively few arcs: we effectively store only the non-zero entries of the adjacency matrix, in a linked list:

Example: Exercise #52, p. 429.



etc.

Use convention that we order the neighbors (in the obvious) – e.g. lowest to highest – for uniqueness.

The redundancy in drawing the adjacency list for an undirected graph is evident. This is eliminated for a directed graph:

Example: Exercise #61, p. 430.