Section 7.1: Boolean Algebra Structure

April 9, 2009

Abstract

First of all, note that we're only reading 7.1 through p. 541 (up to Isomorphic Boolean Algebras).

A Boolean algebra (named after George Boole) is a generalization and an abstraction of the propositional logic we studied early this term, as well as the set theory which we glanced at, albeit briefly. We are really interested in using it to understand the basic elements of computer logic, however, which is based on a binary (0,1) alphabet. In this first section we are merely introduced to the fundamental concepts of Boolean algebra.

1 Definition and Terminology

Definition: a **Boolean Algebra** is a set B on which are defined two binary operations $+$ and $\cdot$, and one unary operation $'$, and in which there are two distinct elements 0 and 1 such that the following properties hold for all $x, y, z \in B$:

$1a.\ x + y = y + x$	$1b.\ x \cdot y = y \cdot x$	commutative property
$2a.\ (x + y) + z = x + (y + z)$	$2b.\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$	associative property
$3a.\ x + (y \cdot z) = (x + y) \cdot (x + z)$	$3b.\ x \cdot (y + z) = (x \cdot y) + (x \cdot z)$	distributive property
$4a.\ x + 0 = x$	$4b.\ x \cdot 1 = x$	identity property
$5a.\ x + x' = 1$	$5b.\ x \cdot x' = 0$	complement property

The element x' is called the **complement** of x. The algebra may be denoted $[B, +, \cdot, ', 0, 1]$.

Of these properties, certainly the distributive property 3a. may seem the strangest, since it obviously doesn't hold for the usual suspects $+$ and $\cdot$.

Notice the beautiful **symmetry** in this definition: the roles of $+$ and $\cdot$ are exactly reversed with respect to the special elements 0 and 1.

Question: how are these reflected in the properties of propositional logic that we studied earlier this term?

- A comment on functions $f : \{T, F\}^n \to \{T, F\}$, which take elements of the Cartisian product $\{T, F\}^n$ into the set $\{T, F\}$ (e.g., implication).
- The difference between $\iff$ and $=$....

$\longrightarrow$

x	y	$f(x,y)$
T	T	T
T	F	F
F	T	T
F	F	T

In Example 2, p. 538, the set $B = \{0, 1\}$ consisting of **only** two elements (so they must be our distinguished elements), and the binary operations of $+$ by $x + y = max(x, y)$ and that of $\cdot$ by $x \cdot y = min(x, y)$. Complements are given by $0' = 1$ and $1' = 0$. It turns out that this is another example of a Boolean algebra.

Example: Practice 1, p. 538 : Verify property 4b for the Boolean algebra of Example 2.

$x \cdot 1 = x$

Proof by exhaustion:

$1 \cdot 1 = 1$

$0 \cdot 1 = 0$

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

1.1 Idempotence

Curiously enough, $x + x = x$ in a Boolean algebra (this is the **idempotent property**. You'll want to remember that one, for any proofs!) And since $x + x = x$, we must have $x \cdot x = x$ by the beautiful symmetry of the operations. This symmetry, known as **duality**, means that we only have to do half the work most of the time....

You may have bumped into this concept in linear algebra: for example, projection matrices are idempotent, such as

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix projects onto the first, third, and fourth dimensions; and projecting onto those dimensions a second time doesn't change anything (i.e., $A \cdot A = A$).

Example: Practice 2, p. 540

a. What does the idempotent property of Example 3 become in the context of propositional logic?

$$A \vee A \iff A$$
$$A \wedge A \iff A$$

b. What does it become in the context of set theory?

$$A \cup A = A$$
$$A \cap A = A$$

Example: Practice 3, p. 540

a. Prove that the property $x + 1 = 1$ holds in any Boolean algebra. Give a reason for each step.

$$\begin{aligned} x+1 &= x + (x + x') && \text{5a} \\ &= (x + x) + x' && \text{associativity} \\ &= x + x' && \text{idempotence} \\ &= 1 && \text{5a} \end{aligned}$$

b. What is the dual property?

$$\boxed{x \cdot 0 = 0}$$

1.2 Complements are Unique

Given an element x of the set B of a Boolean algebra, the complement x' is the **unique** element of B with the property that

$$x + x' = 1 \text{ and } x \cdot x' = 0$$

2 Hints for proving Boolean Algebra Equalities (p. 540)

- Usually the best approach is to start with the more complicated expression and try to show that it reduces to the simpler expression.
- Think of adding some form of 0 (like $x \cdot x'$) or multiplying by some form of 1 (like $x + x'$).
- Don't forget property 3a, the distributive property of addition over multiplication, just because it seems weird!
- Remember the idempotent properties: $x + x = x$ and $x \cdot x = x$. + Uniqueness of Complements
- It may help to frame the argument in terms of either set theory or propositional logic, to give you a framework for understanding the thrust.

3 Examples

Example: Exercise 4b, p. 547 : Prove De Morgan's Laws for any Boolean algebra, e.g.

$$(x + y)' = x' \cdot y'$$

$$\begin{aligned}
&(x+y) + x'\cdot y' \\
&= ((x+y)+x')\cdot((x+y)+y') \\
&= (x+(y+x'))\cdot(x+(y+y')) \\
&= (x+(x'+y))\cdot(x+1) \\
&= ((x+x')+y)\cdot 1 \\
&= (1+y)\cdot 1 \\
&= 1\cdot 1 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
&(x+y)\cdot x'\cdot y' \\
&= x'\cdot y'\cdot(x+y) \\
&= x'\cdot y'\cdot x + x'\cdot y'\cdot y \\
&= x'\cdot x\cdot y' + x'\cdot 0 \\
&= 0\cdot y' + 0 \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

$\therefore$ by the uniqueness of the complement,

$$x'\cdot y' = (x+y)'$$

Example: Exercise 9, p. 548 : Prove that in any Boolean algebra

$$x \cdot y' = 0 \iff x \cdot y = x$$

$\Rightarrow$: assume $x \cdot y' = 0$

$$\begin{aligned} x \cdot y &= xy + 0 \\ &= x \cdot y + x \cdot y' \\ &= x \cdot (y + y') \\ &= x \cdot 1 \\ &= x \checkmark \end{aligned}$$

$\Leftarrow$: assume $x \cdot y = x$

$$\begin{aligned} x \cdot y' &= (x \cdot y) \cdot y' \\ &= x \cdot (y \cdot y') \\ &= x \cdot 0 \\ &= 0 \end{aligned}$$

Example: Exercise 12a, p. 548 : Prove that in any Boolean algebra

$$x + y = 0 \rightarrow x = 0 \wedge y = 0$$

$\Rightarrow$: Assume $x + y = 0$

we know that

$X \cdot X' = 0$

Then

$x + y = x \cdot x'$

$$\begin{aligned} x &= x + 0 \\ &= x + (x + y) \\ &= (x + x) + y \\ &= x + y \\ &= 0 \end{aligned}$$

By symmetry, $y = 0$.