Lab for the Fundamental Theorem of Calculus (part II)

Big picture: defining functions in terms of integrals.

We now know how to evaluate definite integrals $\int_a^b f(x)dx$, if we have formulas for $f(x)$ and know an anti-derivative for f (let's call it F):

$$\int_a^b f(x)dx = F(b) - F(a)$$

So evaluation's easy in this case.

The second part of the Fundamental Theorem of Calculus gives us a very impressive new tool: it allows us to **define new kinds of functions** using integrals. What does this mean?

Consider a definite integral (and notice that we're going to switch x for a new dummy variable of integration, t – this is because we love to use x for our variable in functions):

$$\int_a^b f(t)dt = F(b) - F(a)$$

what's so special about the letter b? Could we just as well write

$$\int_a^x f(t)dt = F(x) - F(a)$$

If so, we can think of x as a variable, and so we've used the definite integral to define a new function, based on $f(t)$.

This function is an anti-derivative of f:

$$F(x) = \int_a^x f(t)dt + F(a)$$

Hence, $F'(x) = f(x)$.

THEOREM 1 Fundamental Theorem of Calculus, Part II Let $f(x)$ be a continuous function on $[a, b]$. Then $A(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$, that is, $A'(x) = f(x)$, or equivalently,

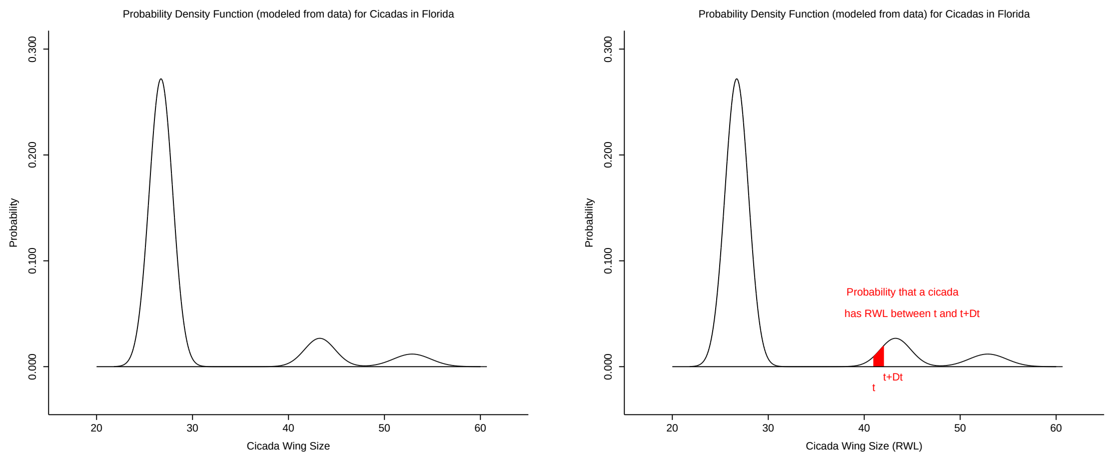
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Furthermore, $A(x)$ satisfies the initial condition $A(a) = 0$.

Of course our author uses A, rather than F, to emphasize that we get an **anti-derivative**. The basic idea is that we can use the integral, which was derived to represent the area under a curve, as a means to creating or representing anti-derivatives for functions.

This is a novel concept: we've been thinking of integrals as representing area under a curve. Now we're going to "liberate" an endpoint – turn it into a variable, instead of thinking of it as a fixed constant, and so the answer becomes a function.

One important application where we see functions defined in terms of integrals is in probability. I've studied cicada-killer wasps, and so we've studied populations of cicadas. The figure below shows the distribution of sizes of cicadas in a particular part of Florida, modeled on real data. At left is the probability density function $\rho(x)$. As a probability



density function it has some important properties, like $\rho(x) \geq 0$ and

$$\int_{-\infty}^{\infty} \rho(x)dx = 1$$

What this says is that every cicada is somewhere: the probability is 1 that you'll find a cicada with wing length somewhere between $-\infty$ and ∞ (big deal, right?! We knew that...).

At right in the first figure we illustrate the probability of finding a cicada in a small band of right wing lengths between t and $t + \Delta t$ (I should have used Δ, but my software won't plot Greek letters in figures!). We can think of this tiny probability (i.e. area) as an integral:

$$\Delta P(t) \equiv P(t \leq x \leq t + \Delta t) = \int_{t}^{t+\Delta t} \rho(x)dx$$

In general, we define the Cumulative Distribution Function P as

$$P(t) = \int_{0}^{t} \rho(x)dx$$

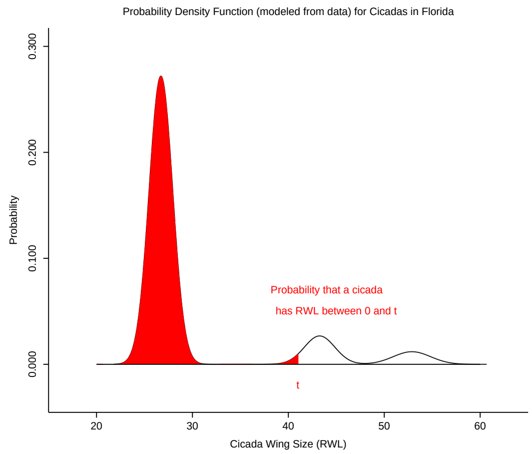
(we can start our lower limit at 0, rather than $-\infty$, because no RWL is negative). In the second set of figures below, we have at left the probability that a cicada has a RWL between 0 and t (i.e., $P(0 \leq x \leq t)$, which is just $P(t)$). At right in the figure is a plot of $\rho(x)$ and $P(x)$ together. One is the density, and the other the cumulative function.

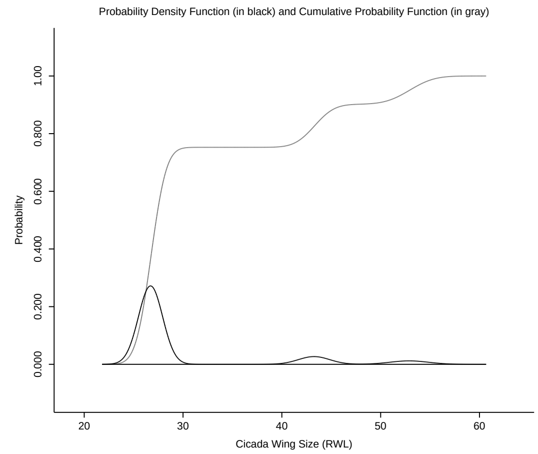
Problem 1: Now, to make the connection elaborated in the FTC II, consider the plot of $P(t \leq x \leq t + \Delta t)$. In the following "equations", insert either $=$ or $\approx$ to make each mathematical phrase correct:

1.

$$\Delta P(t) \equiv P(t + \Delta t) - P(t) \qquad\qquad \Delta t \cdot \rho(t)$$

What does the right-most quantity represent graphically?





2. Relate the following four quantities using $\equiv$, $=$, or $\approx$:

$$P'(t) \qquad \frac{\Delta P(t)}{\Delta t} \qquad \frac{P(t+\Delta t)-P(t)}{\Delta t} \qquad \rho(t)$$

3. Now, in the limit as $\Delta t \longrightarrow 0$, we have

$$P'(t) \qquad \lim_{\Delta t \longrightarrow 0} \frac{\Delta P(t)}{\Delta t} \qquad \lim_{\Delta t \longrightarrow 0} \frac{P(t+\Delta t)-P(t)}{\Delta t} \qquad \rho(t)$$

Problem 2: justify the shapes of the graphs of ρ and P, relative to each other, as seen in the figure above (at right). What should their relationship be?