Section 1.2: Propositional Logic

August 30, 2010

Abstract

Now we're going to use the tools of formal logic to reach logical conclusions ("prove theorems") based on wffs formed by some given statements. This is the domain of propositional logic.

- **Propositional wff**: represent some sort of argument, to be tested, or proven, by **propositional logic**.

- **valid arguments**, e.g.

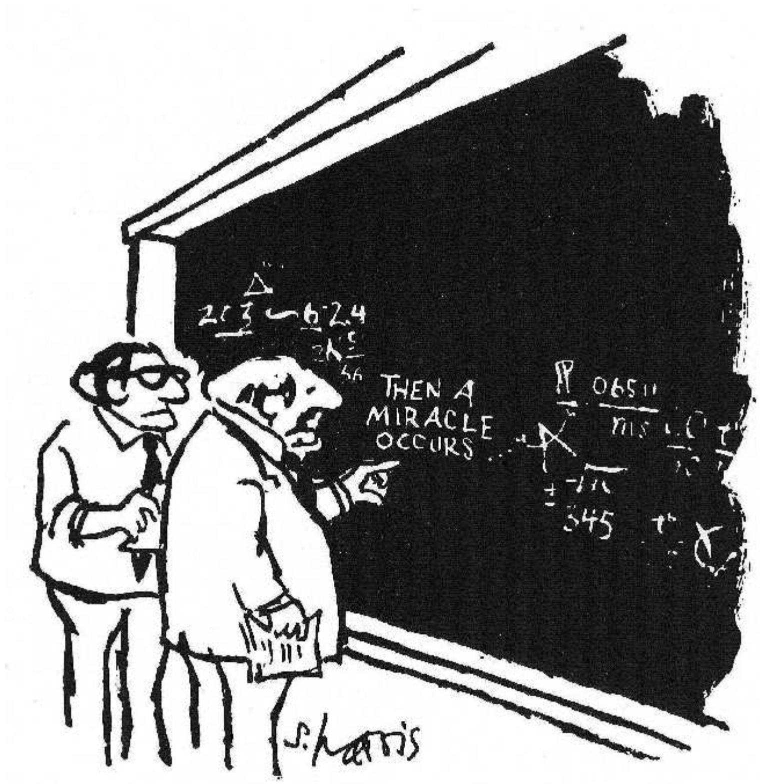
$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \to Q$$

  have **hypotheses** (we suppose – assert – that the P_i are true), and a **conclusion** (Q). To be *valid*, this argument must be a tautology (always true). To be an *argument*, Q must not be identically true (i.e. a fact, in which case the hypotheses would be irrelevant!).

- **Proof Sequence**: a sequence of wffs in which every wff is an hypothesis or the result of applying the formal system's derivation rules (truth-preserving rules) in sequence.

  Our Objective: to reach the conclusion Q from the hypotheses P_1, P_2, ..., P_n.

Sidney Harris knows that you need to follow the rules in a proof sequence:



"I think you should be more explicit here in step two."

- Types of derivation rules:
  - **Equivalence rules** (see Table 1.12, p. 24): we can substitute equivalent wffs in a proof sequence. One way of showing that two wffs are equivalent is via their truth tables.
    * commutative
    * associative
    * De Morgan's laws
    * implication ($P \longrightarrow Q \iff P' \vee Q$)
    * double negation

    Implication seems somewhat unusual, but it is suggested by Exercise 7a, section 1.1: "If the food is good, then the service is excellent."

    Now negate it! This leads to the rule $(P \longrightarrow Q)' \iff P \wedge Q'$.

    You're asked to prove the implication equivalence rule in Practice 9, p. 24. That is, prove that

    $$P \rightarrow Q \longleftrightarrow P' \vee Q$$

    is a tautology (notice how we use order of precedence of operations to stay sane). How would you prove it?

  - **Inference rules**: from given hypotheses, we can deduce certain conclusions (see Table 1.13, p. 25)
    * **modus ponens:** If Q follows from P, and P is true, then so is Q.
      1. P
      2. $P \rightarrow Q$ | 3. Q
    * **modus tollens:** If Q follows from P, and Q is false, then P is also false.
      1. $P \rightarrow Q$ | 3. P'
      2. Q'
    * **conjunction:** If Q is true, and P is true, then they're both true together.
      1. Q
      2. P
      3. $Q \wedge P$

* **simplification:** If both Q and P are true, then they're each true separately.

  1. $Q \wedge P$
  2. Q

* **addition:** If P is true, then either P or Q is true.

  1. P
  2. $P \vee Q$

Of these, addition may seem a little odd: what do you gain by adding an arbitrary argument Q to an already true wff P into a logical or?

Practice 10, p. 26. Also give step 4! + 5!

1.	$(A \vee B') \to C$	*hyp*
2.	C'	*hyp*
3.	$(A \vee B')'$	1, 2, mt
4.	$A' \wedge (B')'$	3, deMorgan
5.	$A' \wedge B$	4, double negation

For a more elaborate example, let's look at #29, p. 33, which shows that one can prove anything if one introduces a contradiction (e.g. #6, on the mensa quiz). Also called an **inconsistency**, this is a beautiful and important example:

$$P \wedge P' \to Q$$

1.	P	hyp
2.	P'	hyp
3.	$P' \vee Q$	2, add.
4.	$P \vee Q$	1, add
5.	$(P \vee Q) \wedge (P' \vee Q)$	3, 4, conj
6.	$(P \wedge P') \vee Q$	5, dist
7.	$P \vee Q'$	1, add
8.	$(P' \wedge Q)'$	7, de Morgan
9.	$P \to Q$	3, imp
10.	Q	1, 9, mp

– The difference between equivalence rules and inference rules is that equivalence rules are bi-directional (work both ways), whereas some inference rules are uni-directional (work in only one direction - e.g., simplification; this is what inference is all about: from this we can infer that, but we cannot necessarily infer this from that!).

Notice that in the table 1.14 (More Inference Rules, p. 33) some rules appear twice (e.g. contraposition): two uni-directionals can make a bi-directional (which makes this an equivalence rule).

Note for your homework: you are not allowed to invoke the rule that you are trying to prove! Notice that the entries in this table are followed by exercise numbers: it is in those exercises that the results are obtained!

– **Deduction method**: if we seek to prove an implication, we can simply add the antecedent of this concluding implication to the hypothesis of the argument, and prove the consequent of the concluding implication:

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow (R \rightarrow S)$$

can be replaced by

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge R \rightarrow S$$

This is really "Exportation" (from Table 1.14) backwards, which says that Exportation is behaving like an equivalence rule (it is: check the truth tables, or construct a proof).

If you're interested in seeing why this rule works, you might try #49, p. 34, but think of it this way: we're interested in assuming that all the P_i are true, and see if we can deduce the implication $R \rightarrow S$. If R is false, then the implication is true. The only potentially problematic case is where R is true, and S is false. Then what we want to know is: given that

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge R$$

are true, is S true?

Exercise #35, p. 33:

$(A' \rightarrow B) \wedge (B \rightarrow C) \wedge (C \rightarrow D) \rightarrow (A' \rightarrow D)$

1.	$A' \rightarrow B$	hyp
2.	$B \rightarrow C$	hyp
3.	$C \rightarrow D$	hyp
4.	$A' \rightarrow C$	1, 2, hs
5.	$A' \rightarrow D$	3, 4, hs ✓

1.	...	
2.	...	
3	...	
4	A'	deduction method
5.	B	1, 4, mp
6.	C	2, 5, mp
7.	D	3, 6, mp ✓

- **Hypothetical syllogism**:

$$(P \to Q) \wedge (Q \to R) \to (P \to R)$$

(and see a whole long list of rules in Table 1.14). This rule might be referred to as **transitivity**.

A new rule is created each time we prove an argument; but we don't want to create so many rules that we keel over under their weight! Keep just a few rules in view, and learn how to use them well....

- Our goal may well be to turn a "real argument" into a symbolic one. This allows us to test whether the argument is sound (that is, that the conclusion follows from the hypotheses).

  Exercise #44, p. 34: If the ad is successful, then the sales volume will go up. Either the ad is successful or the store will close. The sales volume will not go up. Therefore the store will close. (A, S, C)

- The propositional logic system is complete and correct:
  - **complete**: **every** valid argument is provable.
  - **correct**: **only** a valid argument is provable.

  The derivation rules are truth-preserving, so correctness is pretty clear; completeness is not! How can we tell if we can prove every valid argument?!