Section 2.1: Sequences of Real Numbers

October 5, 2011

Abstract

Our author presents sequences as a means of studying phenomena of a dynamic nature, given at discrete units of time which can be counted off by the natural numbers.

The question of limits is then a question about the long-term (asymptotic) behavior of those phenomena.

Definition: sequence: A **sequence** of real numbers is a function from the positive integers into the real numbers.

Definition: converges to the number L, limit, diverge: We say that the sequence of real numbers $\{x_n\}$ **converges to the number** L if, $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that if $n > N(\varepsilon)$, $n \in \mathbb{N}$, then $|x_n - L| < \varepsilon$. We say that L is the **limit** of the sequence $\{x_n\}$, and we write $\lim x_n = L$, or $\{x_n\} \to L$. If a sequence does not converge, then it is said to **diverge**.

Indeed, the question of limits is a question about the long-term (asymptotic) behavior of those phenomena, these sequences. It's a question about what we call the **tail** of the sequence – all those terms located past a certain point, past a certain index value of the natural numbers. This is illustrated quite well in Figure 2-1, p. 37.

Example: #2, p. 46 The sequence $\{a_n\} = \{\frac{1}{n}\}$ is composed of strictly positive terms for all $n \in \mathbb{N}$, with a limit of $L = 0$.

Proof: Certainly the terms are positive, as reprocals of positive numbers. Given an $\varepsilon > 0$. Then we must find $N(\varepsilon)$ such that

$$n > N(\varepsilon) \to |a_n - 0| < \varepsilon$$

That is, that

$$n > N(\varepsilon) \to \frac{1}{n} < \varepsilon$$

Simply choose $N(\varepsilon) > \frac{1}{\varepsilon}$: then

$$n > N(\varepsilon) \to n > \frac{1}{\varepsilon} \to \frac{1}{n} < \varepsilon.$$

Definition: diverge to infinity: A sequence of real numbers $\{x_n\}$ is said to **diverge to infinity** if, $\forall M \in \mathbb{R}$, $\exists N(M) \in \mathbb{N}$ such that if $n > N(M),\ n \in \mathbb{N}$, then $x_n > M$. In this case we write $\lim x_n = \infty$, or $\{x_n\} \to \infty$.

Similarly for **diverge to negative infinity**, in the obvious way.

Example: You might not be surprised to learn that the sequence of multiplicative inverses of the sequence of #2, p. 46 diverges to infinity as that sequence converges to 0: $\{b_n\} = \{n\}$ is composed of strictly positive terms for all $n \in \mathbb{N}$, diverging to ∞.

Theorem 2-1: A sequence of real numbers can converge to at most one number.

Proof: (by contradiction): Suppose that there exist two limits of the sequence $\{x_n\}$, $L \neq M$.

Then $\forall \varepsilon > 0$,

(i) $\exists N_1(\varepsilon) \in \mathbb{N}$ such that $n > N_1(\varepsilon) \to |x_n - L| < \varepsilon$

(ii) $\exists N_2(\varepsilon) \in \mathbb{N}$ such that $n > N_2(\varepsilon) \to |x_n - M| < \varepsilon$

WLOG assume that $M > L$, and consider $\varepsilon^* = M - L > 0$. Take $\varepsilon = \dfrac{\varepsilon^*}{4}$. Then $\exists N_1(\varepsilon), N_2(\varepsilon)$ as described above, and hence

$$n > max(N_1(\varepsilon), N_2(\varepsilon)) \to (|x_n - L| < \varepsilon) \wedge (|x_n - M| < \varepsilon)$$

Thus, for such n, $(x_n < L + \varepsilon) \wedge (x_n > M - \varepsilon)$. But since $M = L + \varepsilon^* = L + 4\varepsilon$, we have

$$L + 3\varepsilon < x_n < L + \varepsilon$$

But this violates the transitivity of $L + 3\varepsilon > L + \varepsilon$. Contradiction. Q.E.D.

Theorem 2-2: The sequence of real numbers $\{a_n\}$ converges to L if and only if $\forall \varepsilon > 0$ all but a finite number of terms of $\{a_n\}$ lie in the interval $(L - \varepsilon, L + \varepsilon)$.

Observe: That is: eventually **every term** lies inside the interval $(L - \varepsilon, L + \varepsilon)$. The tail is confined to this band of the range a width of ε about L. And that will be true no matter how **small** the band.

Definition: bounded sequence: A sequence is **bounded** if the terms of the sequence form a bounded set.

Theorem 2-3: If $\{a_n\}$ is a convergent sequence of real numbers, then the sequence $\{a_n\}$ is bounded.

Theorem 2-4: Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that $\{a_n\} \to a$ and $\{b_n\} \to b$. Then

(i) $\{a_n + b_n\} \to a + b$.

(ii) $\{ca_n\} \to ca$ for any $c \in \mathbb{R}$.

(iii) $\{a_n b_n\} \to ab$.

(iv) If $b \neq 0$ and $b_n \neq 0$ for any $n \in \mathbb{N}$, then $\frac{a_n}{b_n} \to \frac{a}{b}$.

Theorem 2-5:

(i) Suppose that $\{a_n\}$ converges to L, and that $a_n \leq K$ for every n. Then $L \leq K$.

(ii) Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences with $a_n \leq b_n$ for every n. Also suppose that $\{a_n\} \to L$ and $\{b_n\} \to K$. Then $L \leq K$.

(iii) If $\{a_n\}$ and $\{b_n\}$ are sequences with $0 \leq a_n \leq b_n$ for every n and if $\{b_n\} \to 0$, then $\{a_n\} \to 0$. (A pinching theorem.)

(iv) If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences with $a_n \leq b_n \leq c_n$ for every n and if $\{a_n\} \to L$ and $\{c_n\} \to L$, then $\{b_n\} \to L$. (**The** pinching theorem!)

Proof: (exercises #8, 9, p. 47) We prove the theorem by starting with a lemma:

Lemma : Consider a sequence $\{a_n\}$ of all positive elements ($a_n \geq 0$) that converges: $\{a_n\} \to a$. Then $a \geq 0$. The proof is by contradiction. Assume not: then $a < 0$. Consider $\varepsilon = \frac{-a}{2}$. Then $\exists N(\varepsilon) \in \mathbb{N}$ such that $n > N(\varepsilon) \to |a_n - a| < \varepsilon$. But

$$|a_n - a| < \varepsilon \to -\varepsilon < a_n - a < \varepsilon \to a - \varepsilon < a_n < a + \varepsilon$$

and hence

$$a_n < a + \varepsilon = \frac{a}{2} < 0,$$

which contradicts the positivity of the elements a_n. Hence, the limit $a \geq 0$.

We now push on to the proof of (i), by application of Theorem 2-4 and a specially conceived sequence. Suppose that $\{a_n\}$ converges to L, and that $a_n \leq K$ for every n. We want to show that $L \leq K$.

Consider sequence $\{b_n\}$, with $b_n = K - a_n$. Then $\{b_n\}$ satisfies the lemma, since $b_n \geq 0$, and since $\{b_n\} \to K - L$ (by Theorem 2-4)[1], we have that $K - L \geq 0$, or $K \geq L$.

Now on to the proof of (ii): suppose that $\{a_n\}$ and $\{b_n\}$ are sequences with $a_n \leq b_n$ for every n. Also suppose that $\{a_n\} \to L$ and $\{b_n\} \to K$. Then we want to show that $L \leq K$.

Choose $\{c_n\} = \{b_n - a_n\}$. Every term of $\{c_n\}$ is positive, since $a_n \leq b_n$ for all n. Furthermore $\{c_n\}$ converges by Theorem 2-4, with limit $K - L$. Hence, by the lemma, the limit $b - a \geq 0$; that is, $a \leq b$.

[1]We haven't shown that the limit of a constant sequence is the constant. Show it!

Next we show (iii): suppose that $\{a_n\}$ and $\{b_n\}$ are sequences with $0 \leq a_n \leq b_n$ for every n and if $\{b_n\} \to 0$. We want to show that $\{a_n\} \to 0$.

By definition, we have that $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that if $n > N(\varepsilon)$, $n \in \mathbb{N}$, then $|b_n| < \varepsilon$. Since $0 \leq a_n \leq b_n$, we also have that $|a_n| < \varepsilon$. Hence, by definition, $\{a_n\} \to 0$.

It is now easy to demonstrate the general pinching theorem, by referring our theorem back to this result (iii): assume that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences with $a_n \leq b_n \leq c_n$ for every n and that $\{a_n\} \to L$ and $\{c_n\} \to L$.

Merely define the sequences $\{b_n - a_n\}$, $\{c_n - a_n\}$, and invoke (iii).

Definition: monotone increasing: Let $\{a_n\} \to a$ be a sequence of real numbers. If $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$, the sequence is **monotone increasing**. If $a_{n+1} > a_n$ for every $n \in \mathbb{N}$, the sequence is **strictly monotone increasing**.

Observe: The definitions of monotone decreasing and strictly monotone decreasing are the obvious parallels.

Theorem 2-6: A bounded monotone sequence converges.

Corollary 2-6:

(i) A monotone increasing sequence either converges or diverges to ∞.

(ii) A monotone decreasing sequence either converges or diverges to $-\infty$.

Theorem 2-7: Let $A_n = [a_n, b_n]$ be a sequence of intervals such that $A_n \supset A_{n+1}$ for $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty}(b_n - a_n) = 0$. Then there is a real number p for which

$$\cap_{n=1}^{\infty} A_n = \{p\}.$$

Proof: We're expecting that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = p$. Now we'll show that.

Both sequences $\{a_n\}$ and $\{a_n\}$ are bounded (they are bounded by the first set A_1: $a_1 \leq a_{n+1} \leq b_{n+1} \leq b_1$), and both are monotone: hence, by Theorem 2-6, they are both convergent. Since

$$\lim_{n\to\infty}(b_n - a_n) = 0$$

we use Theorem 2-4 to conclude that

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$$

or that $\exists p \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = p$. The proof of Theorem 2-6 shows that the limit of a bounded monotone increasing sequence is the lub of the sequence. (Similarly a decreasing bounded sequence approaches its glb – Problem 13, p. 47).

Now to show that $\forall n \in \mathbb{N},\ p \in A_n$ (by contradiction). Suppose $p \notin A_n$ for some n, and, WLOG, $p < a_n$. Then $p \neq lub$ of monotone increasing sequence $\{a_n\}$, since $p < a_n$. This is a contradiction, so $p \in A_n\ \forall n \in \mathbb{N}$.

Now to show that p is the **only** thing in the infinite intersection (by contradiction). Assume not: that there is another real q in every A_n. WLOG assume that $q < p$. Since $\lim\limits_{n\to\infty} a_n = p$, we know by the definition of limit that $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that if $n > N(\varepsilon)$, then $|a_n - p| < \varepsilon$. Take $\varepsilon = p - q$. Then $\exists N(\varepsilon) \in \mathbb{N}$ such that if $n > N(\varepsilon)$, then $|a_n - p| < p - q$. Written as an inequality, this says that

$$q - p < a_n - p < p - q$$

or that $q < a_n$. That is a contradiction: $q \notin A_n$ and $q \in A_n$. Hence p is the unique element of the infinite intersection.

Definition: nested sets: A sequence of sets $\{A_n\}$ such that $A_n \supset A_{n+1}$ is called a **nested sequence of sets**.

Theorem 2-8: Let A be a nonempty set of real numbers that is bounded above. Then there is a sequence of numbers $\{x_n\}$ such that

(i) $x_n \in A$, $n \in \mathbb{N}$

(ii) $\lim x_n = lub(A)$.

Proof: (outline) If $\alpha \equiv lub(A) \in A$, then the sequence is obvious: choose $x_n = \alpha$, $\forall n$. If $\alpha \equiv lub(A) \notin A$, then, by Theorem 1-16, we know that for every $\varepsilon > 0$, the interval $(\alpha-\varepsilon, \alpha)$ contains an infinite number of points of A. Choose a sequence of ε_n that approach zero (e.g. $\varepsilon_n = \dfrac{1}{n}$), and take one of the (infinite number of) elements of the sequence within ε_n of α.

Observe: An analogous result holds for the $glb(A)$.

Definition: Cauchy sequence: A sequence $\{a_n\}$ is called a **Cauchy sequence** if, $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that if $n, m > N(\varepsilon),\ n, m \in \mathbb{N}$, then

$$|a_n - a_m| < \varepsilon.$$

Theorem 2-9: A sequence converges if and only if it is a Cauchy sequence.

Note: It's not obvious that a Cauchy sequence should imply convergence: it does says that the terms are getting within ε of each other, no matter how small ε, but you might imagine that, although the terms are getting closer together, they're drifting around, not actually approaching a fixed limit. If you did imagine that, you'd be wrong – that's what the theorem says, at any rate!

The proof of this theorem is found in the exercises for section 2.3....