Section 2.2: Subsequences of Real Numbers

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Abstract

Subsequences are sequences, formed from other sequences. We draw the line at subsequences, and don't really require subsubsequences, subsubsubsequences, or even sub^{n}sequences.

Definition: subsequence: Let $\{a_n\}$ be a sequence of real numbers, and let

$$n_1 < n_2 < \cdots < n_k < \cdots$$

be a strictly increasing sequence of natural numbers. Then

$$\{a_{n_1}, a_{n_2}, \cdots, a_{n_k}, \cdots\}$$

is called a **subsequence** of $\{a_n\}$ and is denoted by $\{a_{n_k}\}$.

Note: A subsequence is formed from a sequence by selecting certain terms from the sequence **in order**.

Definition: subsequential limit: Let $\{a_n\}$ be a sequence of real numbers. We say that L is a **subsequential limit** if there is a subsequence of $\{a_n\}$ that converges to L.

Example: Consider the sequence

$$a_n = \begin{cases} \frac{1}{n} & n \text{ even} \\ \pi & n \text{ odd} \end{cases} \tag{1}$$

Then there is a subsequential limit of 0, and subsequential limit of π.

Theorem 2-10: A sequence $\{a_n\}$ converges to L if and only if every subsequence of $\{a_n\}$ converges to L.

Note: So clearly the example sequence (1) defined above has no limit, since there are two distinctly different subsequential limits.

Example: #1, p. 51

(i) $\{\frac{1}{2}, \frac{-1}{2}, \frac{3}{4}, \frac{-3}{4}, \frac{7}{8}, \frac{-7}{8}, \ldots\}$

(ii) $\{\sin\left(\frac{n\pi}{2} + \frac{1}{n}\right)\}$

(iii) $\{[(-1)^n + 1]n\}$

Theorem 2-11: The number L is a subsequential limit point of the sequence $\{a_n\}$ if and only if for any $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many terms of $\{a_n\}$.

Corollary 2-11: Let $\{a_n\}$ be a sequence of real numbers. Then L is a subsequential limit point of the sequence if and only if for any $\varepsilon > 0$ and for any positive integer N, there is a positive integer $n(\varepsilon, N) > N$ for which $|a_{n(\varepsilon,N)} - L| < \varepsilon$.

Note: If you like, we could consider stepping off the terms of a subsequence with limit L by

(i) choosing a sequence of ε_n tending to zero,

(ii) Starting with ε_1 and a challenge N, take $n_1 = n(\varepsilon_1, N)$.

(iii) Iterate, always choose the next "challenge N" to be greater than the preceding n_k.

Example: Exercise 8, p. 51 By considering this exercise we discovered some really fascinating things about sequences and subsequential limits. Lindsay encapsulated it well by saying that a countable sequence (a sequence contains at most a countably infinite number of different values) can approach an uncountable number of subsequential limits. Very strange!

For the countably infinite case, we discovered a sequence whose elements have every natural number as subsequential limits: (1,1,2,1,2,3,1,2,3,4,1,2,3,4,5,....). We discovered that $a_i = 1$ if $i = \frac{k(k+1)}{2} + 1$, for $k \in \mathbb{N}$ (or $k = 0$).

For which values of i do we have $a_i = m$, where m is a natural number? Define $t_k \equiv \frac{k(k+1)}{2}$, the k^{th} triangular number. Then

$$i = t_k + (m-1), k \in \{m-1, m, m+1, m+2, \ldots\}$$

To show that every rational is a subsequential limit, we simply order the rationals by the integers as $\{a_n\}$, then use the same idea for the sequence $\{q_n\}$:

$$(a_1, a_1, a_2, a_1, a_2, a_3, a_1, a_2, a_3, a_4, a_1, a_2, a_3, a_4, a_5, ...)$$

so that we visit every rational number an infinite number of times.

To show that every **real** is a subsequential limit, we must argue that we can take a subset of $\{q_n\}$ to approach any real arbitrarily. Since every rational is repeated an infinite number of times, the idea is illustrated by attempting to find π as a subsequential limit: take the first occurance of 3, then the first occurance of 3.1 following 3, then the first occurance of 3.14 following that, etc.: $(3, 3.1, 3.14, 3.141, 3.1415, \ldots)$. With each iterate we get closer and closer to our real value r with our rational-valued q_n.