Section 2.3: Bolzano-Weierstrass Theorem

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Abstract

The Bolzano-Weierstrass Theorem says something intuitive: that a set of numbers, of infinite cardinality yet whose elements are bounded in size, is going to have to huddle around at least one point. You can't pack that many points into a confined space without leaving a clump of them about some point.

We define a couple of new notions, lim sup and lim inf, related to the limits that we're going to find in such bounded sets with an infinite number of elements.

Definition: limit point: A number x is called a **limit point** (or **cluster point** or **accumulation point**) of a set of real numbers A if, $\forall \varepsilon > 0$, the interval $(x-\varepsilon, x+\varepsilon)$ contains infinitely many points of A.

Theorem 2-12 (Bolzano-Weierstrass): Every bounded infinite set of real numbers has at least one limit point.

Note: Clearly some bounded infinite sets of real numbers have **no more** than one limit point (e.g. the set represented by the sequence $\{2^{-n}\}$) – here thinking of a sequence as representing a set. This is a potential cause for confusion! Be careful.... We're transitioning from sequences to general sets.

Proof: (sketch) Since A is bounded, $\exists M > 0 \;/\; A \subset [-M, M]$. Now cut the interval in half, and choose a half that has an infinite number of elements in it (WLOG $A_1 \equiv [0, M]$). Repeat this process over and over, creating a set of nested intervals as in Theorem 2-7, whose widths tend to zero. Thus there is some p such that $\cap_{n=1}^{\infty} A_n = \{p\}$. Using the intervals created, we can create an appropriate sequence.

Theorem 2-13: Consider the sequence $\{a_n\}$. Then L is a subsequential limit of $\{a_n\}$ if and only if L satisfies either of the following conditions:

(i) There are infinitely many terms of $\{a_n\}$ that equal L.

(ii) L is a limit point of the set consisting of the terms of $\{a_n\}$.

Note: Here we are connecting a result about sequences and their convergence to a result about sets and **cluster** points or **accumulation** points (i.e. **limit** points) – how suggestive these names are!

Proof:

- $\rightarrow$: Let's assume that L is a subsequential limit of $\{a_n\}$. Then by theorem 2-11, $\forall \varepsilon > 0$, $\exists$ infinite number of terms in the interval $(L - \varepsilon, L + \varepsilon)$.

  If there are an infinite number of repeated terms L, then the set $\{a_n\}$ may be a singleton (and hence would have no limit point) – but we've satisfied the first condition of the right-hand side.

  Suppose not, therefore: then there is a subsequence of $\{a_n\}$ that converges to L, call it $\{b_n\}$.

  There can be no other value M such that there are an infinite number of terms M in $\{b_n\}$ – otherwise, M would be a subsequential limit point different from L – a contradiction of convergence to L.

  Hence we can now construct an infinite set of distinctly different points in any $\varepsilon-$neighborhood of L. Given $\varepsilon > 0$. Let $\varepsilon_0 = \varepsilon$. Find an element different from L in the interval $(L - \varepsilon_0, L + \varepsilon_0)$ (call it e_1). Now set $\varepsilon_1 = |e_1 - L|$. Find an element different from L in the interval $(L - \varepsilon_1, L + \varepsilon_1)$ (call it e_2). In this way we will create an infinite set of distinctly different points in the interval $(L - \varepsilon, L + \varepsilon)$, and hence L is a limit point of the terms of $\{a_n\}$.

- $\leftarrow$ We assume that L satisfies either of the following conditions:

  (i) There are infinitely many terms of $\{a_n\}$ that equal L, or

  (ii) L is a limit point of the set consisting of the terms of $\{a_n\}$.

  Assume that there are infinitely many terms of $\{a_n\}$ that equal L. Then choose those elements of the sequence as the subsequence, which converges to L as a constant sequence.

  Hence, assume that there aren't infinitely many terms equal to L , but that L is a limit point of the set consisting of the terms of $\{a_n\}$. Then, by definition $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many points of A. Let's construct a subsequence that converges to L, by induction. Let $\varepsilon_n = \frac{1}{2^n}$ (this is an arbitrary sequence that approaches 0 – I could just as well have chosen $\varepsilon_n = \frac{1}{n}$, for example).

  Let $s_1 \equiv a_{n_1}$ be any element in the interval $(L - \varepsilon_1, L + \varepsilon_1)$ (of which there are infinitely many). Then to choose the next element, we seek any element in the interval $(L - \varepsilon_2, L + \varepsilon_2)$ (of which there are infinitely many); hence we can find the first one past the index n_1, for example, and call that index n_2: $s_2 \equiv a_{n_2}$.

  The inductive proposition is

$$P(k): \qquad s_k \equiv a_{n_k} \in (L - \varepsilon_k, L + \varepsilon_k) \wedge n_k > n_{k-1}$$

Assuming $P(k)$, we need to show $P(k+1)$. Consider the subsequence of $\{a_n\}$ where $n > n_k$. Since $\{a_n\}$ contains an infinite number of elements in the interval $(L - \varepsilon_{k+1}, L + \varepsilon_{k+1}$, so does the subsequence. Choose one such element, at index n_{k+1} ($> n_k$), and set $s_{k+1} \equiv a_{n_{k+1}}$. Then $P(k+1)$.

Hence, we conclude that there is a subsequence, $\{a_{n_k}\}_{k\in\mathbf{N}}$ such that $\forall\varepsilon > 0\ \exists N(\varepsilon)$ such that $n > N(\varepsilon) \to |a_n - L| < \varepsilon$. Given any ε , we simply find the element $\varepsilon_k = \dfrac{1}{2^k} < \varepsilon$, and all the elements of the subsequence beyond the corresponding n_k in the sequence we constructed will be with the $\varepsilon-$neighborhood.

Theorem 2-14: Every bounded sequence has a convergent subsequence.

Corollary 2-14: A bounded sequence that does not converge has more than one subsequential limit point.

Note: We will prove this one as a homework exercise (#17, p. 59).

Theorem 2-15:

(i) A sequence that is unbounded above has a subsequence that diverges to ∞.

(ii) A sequence that is unbounded below has a subsequence that diverges to $-\infty$.

Theorem 2-16: A sequence $\{a_n\}$ converges if and only if it is bounded and has exactly one subsequential limit point.

Definition: lim sup: Let $\{a_n\}$ be a sequence of real numbers. Then $\limsup a_n = \overline{\lim}\ a_n$ is the least upper bound of the set of subsequential limit points of $\{a_n\}$, and $\liminf a_n = \underline{\lim}\ a_n$ is the greatest lower bound of the set of subsequential limit points of $\{a_n\}$.

Note: We show in exercise 16 that the supremum of a set of limit points is a limit point of the sequence, as is the infimum. That means that the $\overline{\lim}\ a_n$ and $\underline{\lim}\ a_n$ are the max and min of the subsequential limit points.

Theorem 2-17: Let $\{a_n\}$ be a *bounded* sequence of real numbers. Then

(i) $\overline{\lim}\ a_n = L$ if and only if, $\forall\varepsilon > 0$, there are infinitely many terms of $\{a_n\}$ in $(L-\varepsilon, L+\varepsilon)$ but only finitely many terms of $\{a_n\}$ with $a_n > L + \varepsilon$.

(ii) $\underline{\lim}\ a_n = K$ if and only if, $\forall\varepsilon > 0$, there are infinitely many terms of $\{a_n\}$ in $(K - \varepsilon, K + \varepsilon)$ but only finitely many terms of $\{a_n\}$ with $a_n < K - \varepsilon$.

Corollary 2-17: A bounded sequence $\{a_n\}$ of numbers converges if and only if

$$\overline{\lim}\ a_n = \underline{\lim}\ a_n$$

Theorem 2-18: Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences. Then

(i) $\overline{\lim}(a_n + b_n) \leq \overline{\lim}\ a_n + \overline{\lim}\ b_n$.

(ii) $\underline{\lim}\ a_n + \underline{\lim}\ b_n \leq \underline{\lim}\ (a_n + b_n)$.

Definition: bounded: We say that a function f is **bounded** if the range of f is a bounded set.

Note: If f is bounded, we denote $lub(R(f))$ by $sup\ f$, and $glb(R(f))$ by $inf\ f$.

Theorem 2-19: Let f and g be bounded functions with the same domain. Then

(i) $\sup(f + g) \leq \sup f + \sup g$

(ii) $\inf f + \inf g \leq \inf(f + g)$

Example: Exercise 13, p. 58 (proof of Theorem 2-9, from section 2.1)