Section 4.1: Limits and Continuity

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Abstract

You're already familiar with continuous functions: they're the ones whose graphs you can "draw without lifting your pencil from the paper."

We connect this notion to their formal definition in terms of limits. We'll be putting the machinary that we've developed in the first three chapters to use in the familiar realm and concepts of calculus.

Definition 4-1: limit: Let f be a function with domain $D(f)$, and suppose that x_0 is a limit point of $D(f)$. The *limit of* f *as* x *approaches* x_0 is L if, given any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $\forall x \in D(f)$

$$0 < |x - x_0| < \delta(\varepsilon) \rightarrow |f(x) - L| < \varepsilon.$$

In this case, we write $\lim\limits_{x \to x_0} = L$.

NB: There is an error in the text – the definition on page 73 has $\varepsilon < 0$! No negative ε! It's (almost) always a small **positive** number....

Observation: So you challenge me with an ε, $\varepsilon > 0$ but "very small", and I have to come up with a $\delta > 0$ which creates a δ-neighborhood of x_0 such that for all the x-values in the δ-neighborhood of x_0, the corresponding function values are within ε of L. That is, if $x \neq x_0$ is within a δ's distance of x_0, then $y = f(x)$ is within an ε's distance of L.

Figure 4-1 illustrates this nicely, as well as another important point: it is not necessary for f to be defined at x_0, or, even if defined there, for $f(x_0)$ to equal L.

One further point: you may be wondering why we keep flitting between δ-neighborhoods and ε-neighborhoods: now you see clearly that δ-neighborhoods are generally reserved for the domain values, and ε-neighborhoods are for the range values. It's a convenient convention. This goes back to our discussion of sequences, when $\{a_n\}$ values would be confined to an ε-neighborhood of a limit point – a_n is the "y-value" of the sequence function, associated with the "x-value" n.

Theorem 4-1: Let f be a function with domain $D(f)$, and suppose that x_0 is a limit point of $D(f)$. Then $\lim\limits_{x \to x_0} f(x) = L$ iff for every sequence $\{x_n\} \subset D(f)$ with $x_n \neq x_0$ for every n and $\lim\ x_n = x_0$, the sequence $\{f(x_n)\}$ converges to L.

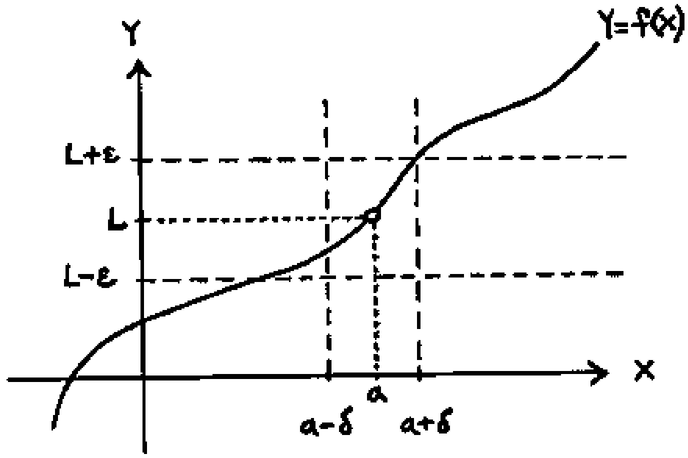

Figure 1: Source

Theorem 4-2: Let f and g be functions with $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$. Suppose h is a function for which $f(x) \le h(x) \le g(x)$ if $x \in (a - \delta, a + \delta) \backslash \{a\}$ for some $\delta > 0$. Then $\lim_{x \to a} h(x)$ exists and $\lim_{x \to a} h(x) = L$.

Definition 4-2: continuous at x_0: Let f be a function and $x_0 \in D(f)$. Then we say that f is *continuous at* x_0 if, given any $\varepsilon > 0$, there is a $\delta(x) > 0$ such that $\forall x \in D(f)$

$$|x - x_0| < \delta(\varepsilon) \to |f(x) - f(x_0)| < \varepsilon.$$

If f is not continuous at x_0, then f is said to be *discontinous* at x_0.

Observation: What's different in this definition (from Definition 4-1) is that we've replace L in the limit by $f(x_0)$, because the function is defined at x_0, and the function values are close to $f(x_0)$ whenever x gets close to x_0. So we'd "fill the hole" in our Figure 1 above.

Corollary 4-1: Let f be a function with domain $D(f)$. Suppose $x_0 \in D(f)$. Then f is continuous at x_0 iff $\lim f(x_n) = f(x_0)$ for every sequence $\{x_n\} \subset D(f)$ with $\lim\ x_n = x_0$.

NB: There is an error in the text – this Corollary is called 4-2 in the text, but is more properly called 4-1 (later references are made to Corollary 4-1).

Theorem 4-3: Let f and g be functions with domains $D(f)$ and $D(g)$, respectively. Suppose $x_0 \in D(f) \cap D(g)$ and both f and g are continuous at x_0. Then

(i) $\alpha f + \beta g$ is continuous at x_0 for any real numbers α and β.

(ii) fg is continuous at x_0.

(iii) f/g is continuous at x_0, if $g(x_0) \neq 0$.

Theorem 4-4: Let f and g be functions with domains $D(f)$ and $D(g)$, respectively, such that the range of f is contained in $D(g)$. If $x_0 \in D(f)$ and if f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Observation: This says roughly that "compositions of continuous functions are continuous" (provided the domains and ranges work together nicely).

Definition 4-3: continuous on a set: Let f be a function with domain $D(f)$ and $A \subset D(f)$. We say that f is *continuous on* A if, given any $\varepsilon > 0$ and $x_0 \in A$, there is a number $\delta(x_0, \varepsilon) > 0$ such that $\forall x \in D(f)$

$$|x - x_0| < \delta(x_0, \varepsilon) \rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Note: δ depends on both x_0 and ε – it's a multivariate function.

Theorem 4-5: Let $f : X \rightarrow Y$. Then f is continuous on $D(f)$ iff for each open set $B \subset Y$, $f^{-1}(B)$ is an open set relative to $D(f)$. That is, $f^{-1}(B) = A \cap D(f)$ where A is an open set in the domain space.

NB: I switched the letters A and B from the text: it seems more proper that, in general, $A \subset X$ and $B \subset Y$.

Theorem 4-6: Let f be a function with domain $D(f)$. The following are equivalent to the condition that f is continuous on $D(f)$.

(i) Given any $\varepsilon > 0$ and $x_0 \in D(f)$, there is a number $\delta(x_0, \varepsilon) > 0$ such that $\forall x \in D(f)$

$$|x - x_0| < \delta(x_0, \varepsilon) \rightarrow |f(x) - f(x_0)| < \varepsilon.$$

(ii) If $x_0 \in D(f)$ and $\{x_n\}$ is any sequence in $D(f)$ with $\{x_n\} \rightarrow x_0$, then $\{f(x_n)\} \rightarrow f(x_0)$; i.e., $f(\lim\ x_n) = \lim\ f(x_n)$.

Note: This one is important: it says that we can "pass the limit inside of the argument of continuous functions".

(iii) If B is an open set of real numbers, then $f^{-1}(B)$ is an open set relative to $D(f)$.

Theorem 4-7: Let $f : X \rightarrow Y$ be a continuous function where X and Y are subsets of $\mathbb{R}$. If A is a compact subset of X, then $f(A)$ is a compact set.

Proof: Given A compact. $f(A) \subset Y$. Suppose $\{U_\alpha\}$ is an open cover of $f(A)$. [The question is: can we reduce it to a finite subcover?] Then $U_\alpha \cap Y$ is open relative to Y.

Since f is continuous, $f^{-1}(U_\alpha)$ is open relative to $D_f = X$ (thm 4-6 (iii)). Given $x_0 \in X$: $f(x_0) \in f(A)$, so $f(x_0) \in U_\alpha$ for some α. Therefore $x_0 \in f^{-1}(U_\alpha)$. Thus $\{f^{-1}(U_\alpha)\}$ is indeed an open cover of A. Consequently there is a finite open subcover of A, call it $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$.

Claim: $\{U_1, \ldots, U_n\}$ covers $f(A)$.

Let $y_0 \in f(A)$. Then $\exists x_0 \in A \;/\; f(x_0) = y$. Since $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$ is a cover of A, $x_0 \in f^{-1}(U_i)$ for some i, so $y_0 = f(x_0) \in U_i$, so every element of $f(A)$ is covered by some U_i.

Therefore $\{U_1, \ldots, U_n\}$ is a finite cover of $f(A)$, and so $f(A)$ is compact. QED.

Corollary 4-7(a): A continuous function on a compact set is bounded.

Proof: $f(A)$ compact $\rightarrow f(A)$ is closed and bounded.

Corollary 4-7(b) (Extreme Value Theorem): A continuous function on a compact set A attains its supremum and infimum on A.

Proof: $f(A)$ compact $\rightarrow f(A)$ closed, $\rightarrow f(A)$ contains all its boundary and limit points (thm 3-6). The sup and inf are either boundary points or limit points (exercise #7, p. 71), hence both are elements of $f(A)$.

Theorem 4-8: If f is a continuous function at $x = c$ and if $f(c) > 0$, then there is a $\delta > 0$ such that $f(x) > 0$ if $x \in (c - \delta, c + \delta) \cap D(f)$.

Corollary 4-8: If f is a continuous function at $x = c$ and if $f(c) < 0$, then there is a $\delta > 0$ such that $f(x) < 0$ if $x \in (c - \delta, c + \delta) \cap D(f)$.

Theorem 4-9: If f is a continuous function on $[a, b]$ and if α is a number between $f(a)$ and $f(b)$, then there is a number $c \in (a, b)$ for which $f(c) = \alpha$.

Corollary 4-9 (Intermediate Value Theorem): If f is a continuous function on $[a, b]$ and if $f(a)$ and $f(b)$ are of opposite signs, then there is a number $c \in (a, b)$ for which $f(c) = 0$.

Definition 4-4: intermediate value property: Let f be a function defined on an interval I. We say that f has the *intermediate value property* if whenever $x_1, x_2 \in I$ with $f(x_1) \neq f(x_2)$, then for any number α between $f(x_1)$ and $f(x_2)$ there is an x_3 between x_1 and x_2 with $f(x_3) = \alpha$.

Definition 4-5: uniformly continuous on A: Let f be a function from a set A to the real numbers. We say that f is *uniformly continuous on* A if, given $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that if $x, y \in A$ and $|x - y| < \delta(\varepsilon)$, then $|f(x) - f(y)| < \varepsilon$.

Note: The only difference between the definition of uniform continuity and continuity is that δ is no longer a multivariate function of x_0 and ε, but rather only a function of ε: δ may be determined independently of which point x_0 we're considering.

Observation: There are cases where δ may be chosen independently of ε! Can you think of one?

Theorem 4-10: If f is a continuous function on a compact set A, then f is uniformly continuous on A.

Corollary 4-10: A continuous function on an interval $[a, b]$ is uniformly continuous.

Example: #24, p. 91 This is a three part problem:

(i) Show that if f is uniformly continuous on a bounded interval I, then f is bounded on I.

Suppose that f is uniformly continuous on I. Then given $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that if $x, y \in A$ and $|x - y| < \delta(\varepsilon)$, then $|f(x) - f(y)| < \varepsilon$.

f must be defined at the midpoint $x = m$ of the interval, which is of width W (since it's bounded). Given $\varepsilon > 0$, and the associated δ. Then we can take steps of size $h = \frac{\delta}{2}$ (say) from the midpoint to each endpoint (there will be $N = ceiling(\frac{W}{\delta})$ of them). As we go, the function can only vary from $f(m)$ by at most ε with each step. That is,

$$|f(x) - f(m)| \leq \sum_{i=1}^{N} |f(m + ih) - f(m + (i-1)h)| \leq N\varepsilon$$

(since the difference between x-values is less than δ) and similarly for the other half of the interval.

$$|f(m) - f(x)| \leq \sum_{i=1}^{N} |f(m - (i-1)h) - f(m - ih)| \leq N\varepsilon$$

Hence we can conclude that $f(m) - N\varepsilon < f(x) < f(m) + N\varepsilon$ for all $x \in I$ – that is, that f is bounded on I.

(ii) Give an example of a continuous function on $(0, 1)$ that is not bounded.

The function $f(x) = \dfrac{1}{x(1-x)}$ is continuous on $(0, 1)$, but certainly unbounded.

(iii) Give an example of a bounded continuous function on $(0, 1)$ that is not uniformly continuous.

The function $f(x) = \sin(\frac{1}{x})$ is continuous on $(0, 1)$, but not uniformly continous. Because it oscillates more and more wildly as x approaches 0, one must take finer and finer δs to assure that the function changes by only ε as one approaches 0.

Example: #25, p. 91 This is a four-part problem:

(i) Show that $f(x) = x$ is uniformly continuous on $\mathbb{R}$.

We can choose $\delta = \varepsilon$, and then conclude that if $x_1, x_2 \in \mathbb{R}$ and $|x_1 - x_2| < \varepsilon$, then $|f(x) - f(y)| = |x_1 - x_2| < \varepsilon$.

(ii) Show that $f(x) = x^2$ is *not* uniformly continuous on $\mathbb{R}$. Thus the product of uniformly continuous functions is not always uniformly continuous.

Consider the righthand side of the definition:

$$|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |(x_1 - x_2)(x_1 + x_2)| = |x_1 - x_2||x_1 + x_2| < \varepsilon$$

If we "solve" for $|x_1 - x_2|$, we find

$$|x_1 - x_2| < \frac{\varepsilon}{|x_1 + x_2|}$$

Hence clearly δ will depend on the sum $x_1 + x_2$. Suppose that f were uniformly continuous. Then, given $\varepsilon > 0$, we would have a constant δ such that

$$|x_1 - x_2| < \delta \to |f(x_1) - f(x_2)| = |x_1 - x_2||x_1 + x_2| < \varepsilon$$

Suppose that $|x_1 - x_2| = \dfrac{\delta}{2} < \delta$, but that $x_1 + x_2 > 2\dfrac{\varepsilon}{\delta}$. Then

$$|x_1 - x_2||x_1 + x_2| = \frac{\delta}{2}|x_1 + x_2| > \frac{\delta}{2}\frac{2\varepsilon}{\delta} = \varepsilon$$

This is a contradiction. Hence $f(x) = x^2$ is *not* uniformly continuous, and so we can see that the product of two uniformly continuous functions (x and x, in this case!) is not necessarily uniformly continuous.

(iii) Show that the sum of two uniformly continuous functions is uniformly continuous.

Given two functions f and g which are uniformly continuous on a set A. Then $f + g$ is uniformly continuous. To show that, we start by recognizing that both f and g satisfy the definition. Hence, given $\varepsilon > 0$, $\exists\delta_f > 0$ and $\exists\delta_g > 0$ such that

$$|x - y| < \delta_f \to |f(x) - f(y)| < \frac{\varepsilon}{2}$$

and

$$|x - y| < \delta_g \to |g(x) - g(y)| < \frac{\varepsilon}{2}$$

Consider the sum function $f + g$:

$$|(f+g)(x)-(f+g)(y)| = |(f(x)+g(x))-(f(y)+g(y))| = |(f(x)-f(y))+(g(x)-g(y))|$$

Continuing,

$$|(f + g)(x) - (f + g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

provided $\delta = min(\delta_f, \delta_g)$. Therefore we have that

$$|x - y| < \delta(\varepsilon) \to |(f + g)(x) - (f + g)(y)| < \varepsilon$$

that is, that $f + g$ is uniformly continuous.

(iv) Show that the product of two uniformly continuous functions on a bounded interval is uniformly continuous.

Given two functions f and g which are uniformly continuous on a bounded interval A. Then fg is uniformly continuous. Both f and g satisfy the definition, hence, given $\varepsilon > 0$, $\exists\delta_f > 0$ and $\exists\delta_g > 0$ such that

$$|x - y| < \delta_f \to |f(x) - f(y)| < \frac{\varepsilon}{M}$$

and

$$|x - y| < \delta_g \to |g(x) - g(y)| < \frac{\varepsilon}{M}$$

where M is any positive number. Consider the product function fg:

$$|(fg)(x)-(fg)(y)| = |f(x)g(x)-f(y)g(y)| = |f(x)g(x)-f(x)g(y)+f(x)g(y)-f(y)g(y)|$$

Thus

$$|(fg)(x) - (fg)(y)| \leq |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|$$

Continuing

$$|(fg)(x) - (fg)(y)| \leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)|$$

and thus

$$|(fg)(x) - (fg)(y)| \leq |f(x)||g(x) - g(y)| + |f(x) - f(y)||g(y)|$$

Since f and g are uniformly continuous on a bounded interval, the functions themselves are bounded – say $|f(x)| < M_f$ and $|g(x)| < M_g$. Then

$$|(fg)(x) - (fg)(y)| \leq M_f|g(x) - g(y)| + M_g|f(x) - f(y)| < M_f\frac{\varepsilon}{M} + M_g\frac{\varepsilon}{M}$$

Taking $M = M_f + M_g$, and with $\delta = min(\delta_f, \delta_g)$, we have that

$$|x - y| < \delta \to |(fg)(x) - (fg)(y)| < (M_f + M_g)\frac{\varepsilon}{M} = \varepsilon$$

Hence the product of two uniformly continuous functions on a bounded interval is uniformly continuous.