Section 5.1: The Derivative

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Abstract

We introduce differentiation and the derivative via the $\varepsilon - \delta$ definition. The usual properties of the derivative are introduced and proven (e.g. derivative of a product, chain rule, etc.).

Definition 5-1: differentiable, derivative: Let f be a function defined on an interval (a, b), and suppose $c \in (a, b)$. We say that f is **differentiable at** c if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite. If this is the case, this limit is called the **derivative of** f **at** c, and is denoted $f'(c)$.

Definition 5-2: (alternative definition of derivative): Let f be a function defined on an interval (a, b), and suppose $c \in (a, b)$. We say that the real number L is the **derivative of** f **at** c if, given $\varepsilon > 0$, there is a number $\delta(\varepsilon) > 0$ such that if $0 < |x - c| < \delta(\varepsilon)$, then

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon.$$

Theorem 5-1: Let f be a function defined on (a, b) with $c \in (a, b)$. Then f is differentiable at $x = c$ with derivative $f'(c)$ if and only if, for every sequence $\{x_n\} \subset D(f)$ with $\{x_n\} \to c$ and $x_n \neq c$ for all n,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c).$$

Theorem 5-2: Let f be a function defined on (a, b), and suppose f is differentiable at $c \in (a, b)$. Then f is continuous at c.

Theorem 5-3: Let f and g be functions defined on (a, b), and differentiable at $c \in (a, b)$. Then

(i) $(\alpha f + \beta g)(c) = \alpha f'(c) + \beta g'(c)$ for any real numbers α and β.

(ii) $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.

(iii) $\left(\dfrac{f}{g}\right)'(c) = \dfrac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$ if $g(c) \neq 0$.

Theorem 5-2 (Chain Rule): Let f be a function defined on (a,b), and suppose $f'(c)$ exists for some $c \in (a,b)$. Suppose g is defined on an open interval containing the range of f, and suppose that g is differentiable at $f(c)$. Then $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Definition 5-3: relative maximum/minimum: A function f has a **relative maximum (minimum)** at a point $c \in D(f)$ if there is a number $\delta > 0$ such that $f(c) \geq f(x)$ ($f(c) \leq f(x)$) for every

$$x \in (c - \delta, c + \delta) \cap D(f).$$

A **relative extremum** is either a relative maximum or minimum.

Theorem 5-5: Let f be a function defined on (a,b), and suppose that f is differentiable at $c \in (a,b)$ and $f'(c) > 0$. Then there is a number $\delta > 0$ such that $f(c) < f(x)$ if $x \in (c, c+\delta)$ and $f(x) < f(c)$ if $x \in (c - \delta, c)$.

Corollary 5-5: Let f be a function defined on (a,b). Suppose that f is differentiable at $c \in (a,b)$ and $f'(c) < 0$. Then there is a number $\delta > 0$ such that $f(c) < f(x)$ if $x \in (c - \delta, c)$ and $f(c) > f(x)$ if $x \in (c, c + \delta)$.

Theorem 5-6: Suppose f is defined on (a,b) and has a relative extremum at $c \in (a,b)$. If f is differentiable at c, then $f'(c) = 0$.