Section Summary: Polar Coordinates

a. Definitions

The **polar coordinate system** (introduced by Newton) is an alternative to the Cartesian coordinate system (named after Descartes) in which every point in the plane is expressed by its distance and direction (angle) from the origin, called the **pole**. The **polar axis** plays the role formerly played by the positive x-axis. The polar coordinates are often given as (r, θ), where

- r is the distance from the pole, and
- θ is the angle between the polar axis and the ray passing through the point of interest.

The **graph of a polar equation** contains all points P that have at least one polar representation (r, θ) whose coordinates solve the equation.

b. **Theorems**

None to speak of.

c. Properties/Tricks/Hints/Etc.

Given r and θ it's easy to find the corresponding x and y:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

It's not quite so easy to go in the opposite direction, unless we restrict r and θ:

$$r \in [0, \infty)$$
$$\theta \in [0, 2\pi)$$

gives a unique representation in polar coordinates (except for the pole, which has equation $r = 0$ regardless of the value of θ).

As a convenience we let r take negative values: so

$$(-r, \theta) = (r, \theta + \pi)$$

In any event, we see that, by contrast with the Cartesian coordinate system, the polar coordinate system allows points to have multiple representations.

To find the Cartesian coordinates from the polar coordinates, we can use the equations

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Be careful, however, as there are two values of θ that solve these equations in each interval of 2π, and one must choose the proper value based on the quadrant in which (x, y) lies....

To plot a polar equation, we may substitute into the equations for x and y:

$$x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta$$

and treat this as an ordinary parametric equation.

d. Summary

The Cartesian coordinate system is very nice for working with horizontal and vertical lines. It is very easy to represent them in Cartesian coordinates:

- vertical lines have the form $x = a$, and
- horizontal lines have the form $y = b$.

In fact, the coordinate axes are, in fact, examples of each of these ($x = 0$ and $y = 0$).

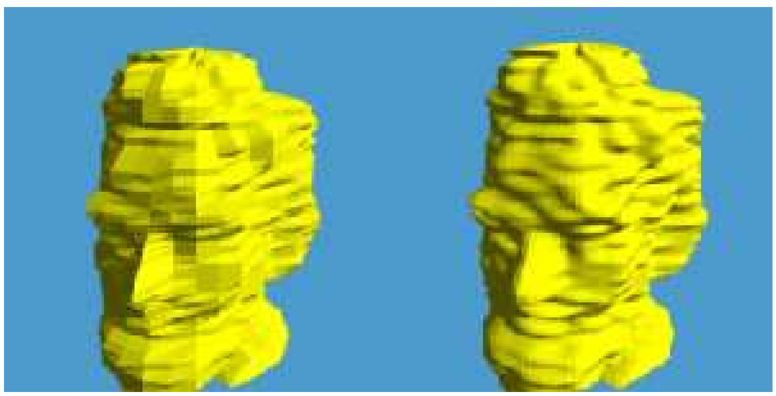
Similarly, the objects most easily represented in polar coordinates would have the form

- $r = a$, or

- $\theta = b$

The first is the form of a circle centered at the origin, whereas the second is the equation of a ray shooting out from the origin, or pole. Notice that when one of these circles and one of these rays intersect, they do so at right angles (just like the coordinate axes in the Cartesian coordinate system).

If you look at drumheads being played, you'll see why this representation might be very useful: there is a lot of circular symmetry, and action along lines eminating from the origin. Situations of this type are often better treated in polar, rather than Cartesian, coordinates.



As another example, my dad and I worked on a bust of Abe Lincoln, using cylindrical coordinates in three dimensions (and polar coordinates are two of the three coordinates used – the third is the z-coordinate). This was appropriate because the data to make the head was taken by measuring the distance of each point on the head from a central axis, as well as the angle from a fixed ray eminating from that axis.

Just as in Cartesian coordinates, where equations are given in the form

$$y = f(x)$$

(one coordinate in terms of the other), in polar coordinates equations often take the form

$$r = f(\theta)$$