Section Summary: Improper Integrals

1 Definitions

Type I Improper Integral:

a. If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided the limit exists as a number.

b. If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided the limit exists as a number. In either case above, the improper integrals are called **convergent** if the corresponding limits exist, and **divergent** otherwise.

c. If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

where the choice of a is completely arbitrary.

Type II Improper Integral:

a. If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

provided the limit exists as a number.

b. If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

provided the limit exists as a number. In either case above, the improper integrals are called **convergent** if the corresponding limits exist, and **divergent** otherwise.

c. If f has a discontinuity at c, where $a < c < b$, and if both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

2 Theorems

$$\int_1^\infty \frac{dx}{x^p}$$

is convergent if $p > 1$ and divergent if $p \leq 1$.

Comparison theorem: Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

a. If $\int_a^\infty f(x)dx$ is convergent, then so is $\int_a^\infty g(x)dx$.

b. If $\int_a^\infty g(x)dx$ is divergent, then so is $\int_a^\infty f(x)dx$.

3 Properties, Hints, etc.

It is easy to be tricked into trying to integrate over a discontinuity: that's the one that's easiest to miss. Watch out! For example, what's troublesome about the following integral?

$$\int_{-1}^1 \frac{1}{x}dx$$

How do limits figure into the calculation of integrals like the one above?

4 Summary

In this section we encounter another strange fact about infinity: though a region may be unbounded, its area may be finite. This can happen in (at least) two different ways: either the domain may head off to $\pm\infty$, or the function may have a vertical asymptote. Both of these cases give rise to what are called "improper" integrals. How delicate sounding!

Both cases are made concrete, are defined, through the use of limits. We see that "partial areas" are used, and if the limits exist, we say that the improper integrals exist.

We can use comparison to determine if an improper integral makes sense: if one unbounded region is contained within another, and the larger is the area corresponding to a convergent improper integral, then the first region's area is finite (its improper integral converges).