Section Summary: Area of a Surface of Revolution

(I) Definitions

The **surface area** S of the surface obtained by rotating the curve $y = f(x)$, with f positive and with a continuous derivative, $a \leq x \leq b$, about the x axis is defined as

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx$$

Writing this formula using the notation given for arc length in section 8.1, we find that

$$S = \int 2\pi y ds$$

which we can think of S as the sum of belts of rectangular area, of thickness ds and circumference $2\pi y$.

(II) Summary

We start with cones, or rather with the frustum of a cone. A surface revolved about an axis can be approximated as a bunch of "frustra", and as they get more and more numerous we might assume that the area of the approximation will converge to the area of the surface itself (the usual trick!).

If you unwrap a cone of radius r and height $\sqrt{l^2 - r^2}$ (that is, with "hypotenus side length" l, as in Figure 2), and lay it flat, then we note a couple of things:

(a) arc length L is a linear function of θ, and

(b) so is area A.

Solving for arc length L using the data points,

$$\begin{bmatrix} \theta & L \\ 0 & 0 \\ 2\pi & 2\pi l \end{bmatrix}$$

we obtain $L = \theta l$.

Solving for area A using the data points,

$$\begin{bmatrix} \theta & A \\ 0 & 0 \\ 2\pi & \pi l^2 \end{bmatrix}$$

we obtain $A = \frac{1}{2}\theta l^2$.

Hence, for the flattened cone,

$$A = \frac{1}{2}\frac{2\pi r}{l}l^2 = \pi r l$$

Now, for a frustum, you take a large cone's area and subtract off a smaller cone's area, leaving the frustrum's area:

$$\Delta A = A_1 - A_2$$

or

$$\Delta A = \pi r_2 l_2 - \pi r_1 l_1 = \pi(r_2 l_2 - r_1 l_1)$$

or

$$\Delta A = \pi(r_2(l_1 + \Delta s) - r_1(l_2 - \Delta s)) = \pi(\Delta s(r_2 + r_1) + (r_2 l_1 - r_1 l_2))$$

which is equal to

$$\Delta A = \pi \Delta s(r_2 + r_1)$$

since

$$r_2 l_1 - r_1 l_2 = 0$$

by "similar cones"!

If we define $R = \frac{1}{2}(r_2 + r_1)$ as the average radius, then we can write

$$\Delta A = 2\pi R \Delta s$$

Now

$$S = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta A_i$$

so

$$S = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi R_i \Delta s$$

which becomes

$$S = \int 2\pi y ds$$

in the limit, where $y > 0$ and ds is the differential chunk of arc length.