Newton's Law of Cooling

"A cup of coffee cooling on the counter, a cake warming in the oven, and a body found in the chill autumn weather...are these the ingredients for a murder mystery to read by the fire or a case for that most famous detective of natural phenomena, Sir Isaac Newton? We use a first-order linear differential equation formulated by Newton to predict the temperatures of objects introduced into media with known ambient temperatures." (Source)

This great lead-in for our lab introduces us to some of the applications of a model Isaac Newton formulated for the cooling (or heating, for that matter) of a "body" which passively responds to ambient temperature. The very basic differential equation we will use to model this cooling is the following:

$$\frac{dT}{dt} = k(A - T) \tag{1}$$

where T is the temperature of the body, A is the ambient temperature, and $k > 0$ is a rate constant (you can see that its units have to be 1/time).

1. If $T > A$ is the body cooling down, or heating up? [Hint: what is the sign of $\frac{dT}{dt}$?]

2. For what value of T is the body in equilibrium (that is, its temperature is unchanging)? [Hint: what will $\frac{dT}{dt}$ be in this case?]

3. If we denote the initial temperature as $T(0) = T_0$, show that the solution of equation (1) is

$$T(t) = A + (T_0 - A)e^{-kt}.$$

4. Coroners use several methods to determine time of death. If equation (1) were used, measurements of the temperature at two different times would be required to establish k and the constant of integration. Suppose this were the only method used to determine time of death in a case where the time of death was the crucial element in the prosecution's case. How would you, as the scientific consultant, help the defense cast doubt on this estimate? Think carefully about the assumptions of the model!

5. From `http://www.biology.arizona.edu/biomath/tutorials/applications/Cooling.html`: We would like to know the time at which a person died. In particular, we know the investigator arrived on the scene at 10:23 pm, which we will call τ hours after death. At 10:23 (i.e. τ hours after death), the temperature of the body was found to be 80^oF. One hour later, $\tau + 1$ hours after death, the body was found to be 78.5^oF. Our known constants for this problem are $A = 68^oF$ and $T_0 = 98.6^oF$.

   At what time did our victim die?

6. Given the following data, produce a reasonable cooling model: Determine the equilibrium

Table 1: "Rectal cooling of the body was studied in 418 normal adult subjects aged between 18-60 years who were admitted and died as a result of traffic accidents at Nehru Hospital of Post Graduate Institute of Medical Education and Research, Chandigarh (India) and on whom postmortems were conducted by the department of Forensic Medicine." Assume an initial temperature of $98.75^o F$. Source: `http://medind.nic.in/jal/t05/i3/jalt05i3p170.pdf`

Time	2	4	6	8	10	12	14	16	18	20	22	24
Temp	98.04	95.07	91.60	88.28	85.33	82.6	80.16	77.96	76.03	74.39	73.05	72.05

temperature and the rate. Plot your model and the data, and see if it looks good or not! Alternatively, you could plot the data and "eyeball" the best fitting exponential curve to determine the parameters of a model....

7. Some (e.g. Henssge and Madea, "Estimation of the time since death in the early post-mortem period") have suggested that Newton's method doesn't get the early cooling period right, and that there's a "sigmoidal" shape to the cooling, as in Figure 1: They propose a model **solution** of

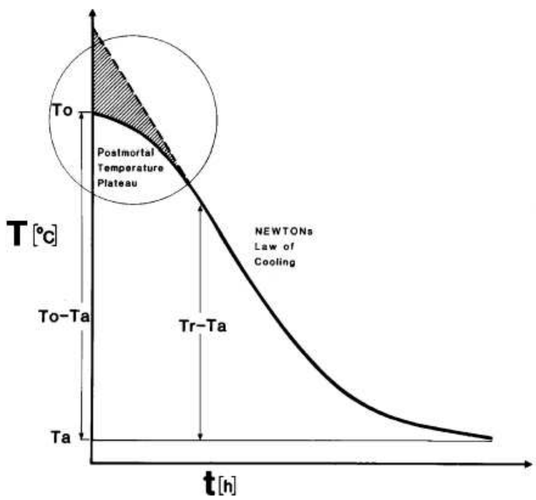

Fig. 1. Sigmoidal shape of body cooling. Mathematical description with a two-exponential-model by Marshall and Hoare [9].

the form

$$T = A + (T_0 - A)\frac{pe^{-kt} - ke^{-pt}}{p - k}$$

where $p > 0$ is another rate constant, which dictates the slower cooling of the body during the first few hours actually seen in data. In order to get the sigmoidal shape, $p > k$.

The India death data was modelled with such a function, and the values obtained for the parameters were

- $T_0 = 99.43$
- $A = 64.11$
- $k = .0720$
- $p = .3679$

Plot the data against the model with these parameters, and judge the fit.