Section Summary: Sequences

1 Definitions

Formally, a **sequence** is a function from the natural numbers ($\mathbb{N}$) into the reals ($\Re$), $a : \mathbb{N} \to \Re$, given explicitly (by showing the terms), by a formula, or by a graph. Informally, a **sequence** is a list of numbers written in a definite order: e.g. $\{a_1, a_2, a_3, \ldots, a_i, \ldots\}$ The numbers a_i are called **terms**, and a_i is called the i^{th} term. Notice that it appears that our sequence just keeps on going: often our sequences will have terms corresponding to every natural number (the natural numbers themselves form a sequence!), and so are infinite in extent (countably infinite, like the natural numbers).

One famous sequence is the **Fibonacci sequence**, which is defined recursively by naming the first two terms in the sequence explicitly and then describing a pattern for the rest of the terms:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= a_{n-1} + a_{n-2} \quad n \geq 3 \end{aligned}$$

This sequence appears throughout nature, and was first described by the 13th century Italian mathematician Fibonacci Pisano.

We'll be interested especially in the behavior of sequences as the terms head off to infinity. A sequence $\{a_n\}$ has **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \;\text{ as }\; n \to \infty$$

if for every $\epsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \epsilon \text{ whenever } n > N$$

If $\lim_{n\to\infty} a_n$ exists, then we say the sequence **converges**, or is **convergent**. Otherwise we say the sequence **diverges**, or is **divergent**.

Sometimes the sequence does not settle down to a limit, but diverges to ∞: $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$a_n > M \quad \text{whenever} \quad n > N.$$

A sequence $\{a_n\}$ is called **increasing** if $a_n \leq a_{n+1}$ for all $n \geq 1$ (that is, $a_1 \leq a_2 \leq a_3 \leq \ldots$. It is called **decreasing** if $a_n \geq a_{n+1}$ for all $n \geq 1$ (that is, $a_1 \geq a_2 \geq a_3 \geq \ldots$. It is called **monotonic** if it is either increasing or decreasing.

A sequence $\{a_n\}$ is called **bounded above** if there is a number M such that

$$a_n \leq M \qquad \text{for all } n \geq 1$$

It is called **bounded below** if there is a number m such that

$$m \leq a_n \qquad \text{for all } n \geq 1$$

If it is bounded both above and below, then it is a **bounded sequence**.

2 Axioms

Completeness Axiom: if S is a non-empty set of real numbers that has an upper bound M (that is, $x \leq M$ for all $x \in S$), then S has a **least upper bound** b (b is an upper bound, and for every upper bound M, $b \leq M$).

3 Theorems

If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is a natural number, then $\lim\limits_{n\to\infty} a_n = L$.

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\begin{array}{l}
\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \\
\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n \\
\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \\
\lim_{n\to\infty}(a_nb_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n \\
\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if} \quad \lim_{n\to\infty} b_n \neq 0 \\
\lim_{n\to\infty} c = c
\end{array}$$

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r: in particular,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & -1 < r < 1 \\ 1 & r = 1 \end{cases}$$

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

4 Properties, Hints, etc.

5 Summary

Sequences are like the natural numbers (1, 2, 3, ...): they have distinct ordered terms traipsing off into the far distance. We're interested in what happens as the terms traipse off. Do they approach a fixed value? Do they oscillate, bouncing back and forth? Do they get larger and larger, or smaller and smaller? Several interesting example are included, such as the Fibonacci numbers (which came about from a silly rabbit population story problem in Fibonacci's *Liber Abaci*).

In this section we encounter many definitions, and a few theorems which help us to understand when a sequence converges (its terms approach a fixed value), or diverges (doesn't converge!). This is an issue of fundamental importance as we push on to our major objective: representing a function using an infinite sequence of functions. We start with numbers, of course, because that's a simpler case.

There are some interesting tricks which rely on our calculus background: using smooth functions to interpolate "the data" (the terms of the sequence), and limits of those functions at infinity.