Section Summary: 6.1 - Inverse functions

- **one-to-one function**: A function f is called a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever} \qquad x_1 \neq x_2$$

  This means that each point in the **range** has a unique **pre-image** (element of the **domain**). Alternatively (and sometimes more usefully),

$$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2.$$

  Let's consider a practical example, e.g. $f(x) = x^2$. This function is not one-to-one, because one element of the range (e.g. 4) has two distinctly different pre-images (-2 and 2). So when someone asks "What's the square root of 4?", the smart-alecky kid (maybe you are one!:) says "-2." Other kids say "well, there are actually two of them...", and another kid says "Our convention is that we say '2', even though we know that there are two different numbers whose squares are 4."

  With one-to-one functions, you never have to have these arguments or discussions or conversations. So the function $f(x) = x^3$ doesn't suffer this fate. If we ask "What's the cube root of 8?", even the smart-alecky kid has to say "2."

- **inverse function**: Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \text{ if and only if } f(x) = y$$

  for any y in B.

- This definition leads to the following cancellation equations:

$$f^{-1}(f(x)) = x \qquad \text{for every } x \in A$$

$$f\left(f^{-1}(y)\right) = y \qquad \text{for every } y \in B$$

- A one-to-one function satisfies the **horizontal line test** (you remember that the vertical line test is one that determines whether a function is, indeed, a function): no horizontal line cuts the graph in two places.

- From the graphical perspective, then, the graph of the inverse function of f (which we can define from a one-to-one function) is obtained by reflecting the graph of f about the line $y = x$. Notice that the **horizontal** line test turns into a **vertical** line test when reflected in the mirror. So if a function passes the horizontal line test, then its reflection will pass a vertical line test, and will consequently be the graph of a function – the inverse function!

- The derivative of the inverse function f^{-1} is given by

  $$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

  This result will be useful as we define the derivatives of inverses of important functions such as exponential functions. Try it out with something easy like the inverse of $y = f(x) = x^3$.

  Perhaps it should be obvious that we can describe the derivative of the inverse function from the derivative of f and its functions values: the slopes of tangent lines of the inverse function are just reflections in the mirror of those of the function itself.

- One of my favorite examples of functions are vending machines. In what ways do pop machines make use of invertible functions, and in what ways do they make use of non-invertible functions?

  Think about this, for example: for a standard pop machine (e.g. one of NKU's Pepsi machines), I push a button and out pops a soda. If you see me pull a regular Pepsi from the machine, can you determine which **button** I pushed?