Name: ______________________

Complete the following problems, showing all your work. P_n denotes all polynomials of degree n or lower, and $M_{m\times n}$ denotes all matrices with m rows and n columns.

You **may** skip up to 10 points: please write "Skip" on the 10 points you skip. (You may "assemble" the 10 points across different problems – 5 from here, 5 from there....)

1. (10 pts) For each of the following spaces, determine a set of vectors that spans the space:

a. $\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} | x + y - 2z = 0\} \subset \mathbb{R}^3$

b. $\{a + bx + cx^2 + dx^3 | a - 2d = 0, b + c = 0\} \subset P_3$

2. (15 pts) Consider the matrix $A = \begin{pmatrix} 4 & -6 & 3 \\ -1 & 3 & 1 \\ 0 & 6 & 7 \end{pmatrix}$ and the vector $\vec{v} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$.

   a. (5 pts) Find the dimension of, and a basis for, the row space of A.

   b. (5 pts) Suppose A represents the coefficients of a homogeneous system of linear equations: e.g., the first equation is $4x - 6y + 3z = 0$. How many solutions are there to this system?

   c. (5 pts) Determine whether $\vec{v}$ is in the column-space of A or not.

3. (10 pts) In each case below determine if the set of vectors S under the inherited operations is a subspace of the vector space V. If it is, prove it; if it is not, then explain why not.

a. (5 pts) $S = \{\begin{pmatrix} a & b \\ a & b \end{pmatrix} | a, b \in \mathbb{R}\}$, $V = M_{2\times 2}$

b. (5 pts) $S = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} | x + z = 1\}$, $V = \mathbb{R}^3$

4. (15 pts) **Use Gauss-Jordan reduction** to find

a. (6 pts) a basis for the row-space,

b. (6 pts) a basis for the column-space, and

c. (3 pts) the rank

of the coefficient matrix of the following system:

$$\begin{aligned} a + b - c + d &= 4 \\ b + c + d &= 2 \\ a + 3b + c + 4d &= 13 \end{aligned}$$

5. (10 pts)

a. (8 pts) If $S = \{\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}\} \subset M_{2\times 2}$, determine if S is linearly independent or dependent in $M_{2\times 2}$.

b. (4 pts) Can the set S serve as a basis for $M_{2\times 2}$? Why or why not?

6. (10 pts)

a. (5 pts) Show that $B = \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \rangle$ is a basis of $\mathbb{R}^2$.

b. (5 pts) Determine the representation of the following vector in this basis: $\mathrm{Rep}_B(\begin{pmatrix} 7 \\ -6 \end{pmatrix})$.

7. (10 pts) If $S_1 = \{\vec{v}, \vec{w}\}$ is a linearly independent set in a vector space V,

a. (5 pts) prove that $S_2 = \{\vec{v} + \vec{w}, \vec{v} + 2\vec{w}\}$ is also a linearly independent set in V.

b. (5 pts) What condition on a and b in the combination $S_3 = \{\vec{v}, a\vec{v} + b\vec{w}\}$ will guarantee a linearly independent set in V?