Name: ______________________

Complete the following problems, showing all your work. P_n denotes all polynomials of degree n or lower, and $M_{m\times n}$ denotes all matrices with m rows and n columns.

Attempt all problems. Partial credit is possible for reasonable observations and approaches.

1. (12 pts: 4 pts each) Let $M = \begin{pmatrix} -3 & 1 \\ 2 & 5 \end{pmatrix}$, $N = \begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 2 \end{pmatrix}$.

Determine or specify "not defined" for each of the following:

a. MN

b. NM

c. M^{-1}

d. N^{-1}

2. (25 pts; 5 pts each) For each item described below, either provide an example of the requested object or state that such an example cannot be found. Justify each answer.

a. an isomorphism from P_4 to $\mathbb{R}^4$.

b. an element of M_{4x4} which is invertible.

c. a map from $\mathbb{R}^4$ to M_{2x2} which is **not** a vector space homomorphism.

d. the rank of a homomorphism from $\mathbb{R}^5$ to $\mathbb{R}^{11}$ which is one-to-one.

e. a homomorphism from $\mathbb{R}^2$ to M_{2x2} which is onto.

3. (21 pts) Let $f : P_3 \to M_{2x2}$ be given by $f(a + bx + cx^2 + dx^3) = \begin{pmatrix} a+d & 0 \\ 0 & b-c \end{pmatrix}$.

a. (3 pts) State how you would prove that f is a homomorphism. (You need not provide the details of the proof.)

b. (7 pts) Determine the nullspace and nullity of f and specify a basis for the nullspace.

c. (7 pts) Determine the rangespace and rank of f and specify a basis for the rangespace.

d. (4 pts) Is f one-to-one? Is f onto? Why or why not?

4. (15 pts) Suppose that $g : \mathbb{R}^m \to \mathbb{R}^n$ is a homomorphism represented with respect to the standard bases by $G = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ -2 & 1 & 0 & 2 & 1 \\ 0 & 5 & 0 & 0 & 1 \end{pmatrix}$.

a. (3 pts) Specify m and n. Is g an isomorphism? Why or why not?

b. (7 pts) Determine the rank of g and the nullity of g and justify your answers.

c. (5 pts) Is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ in the rangespace of g? Justify your answer.

5. (19 pts) Let $B = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$ be the standard basis for $\mathbb{R}^3$ and $D = \langle 1, x, x^2 \rangle$ be the standard basis for P_2. Suppose $h : \mathbb{R}^3 \to P_2$ is the homomorphism whose matrix representation with respect to these bases is given by

$$\text{Rep}_{B,D}(h) = \begin{pmatrix} 1 & -1 & 4 \\ -1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}.$$

a. (4 pts) Is h an isomorphism? Justify your answer.

b. (5 pts) Determine $h(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix})$.

c. (5 pts) If $E = \langle x, x^2, 1 \rangle$ is another basis for P_2, then find $\text{Rep}_{B,E}(h)$.

d. (5 pts) If $C = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rangle$ is another basis for $\mathbb{R}^3$, then find $\text{Rep}_{C,D}(h)$.

6. (8 pts) If $h : V \to W$ is a vector space homomorphism, then prove that h is one-to-one if and only if $\text{nullspace}(h) = \{\vec{0}_V\}$.