11 Fibonacci numbers

In *The Da Vinci Code*, the author Dan Brown made his murdered curator Jacques Saunière leave behind the first eight terms of a sequence of numbers as a clue to his fate. It required the skills of cryptographer Sophie Neveu to reassemble the numbers 13, 3, 2, 21, 1, 1, 8 and 5 to see their significance. Welcome to the most famous sequence of numbers in all of mathematics.

The Fibonacci sequence of whole numbers is:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, . . .

The sequence is widely known for its many intriguing properties. The most basic – indeed the characteristic feature which defines them – is that every term is the addition of the previous two. For example 8 = 5 + 3, 13 = 8 + 5, . . ., 2584 = 1587 + 987, and so on. All you have to remember is to begin with the two numbers 1 and 1 and you can generate the rest of the sequence on the spot. The Fibonacci sequence is found in nature as the number of spirals formed from the number of seeds in the spirals in sunflowers (for example, 34 in one direction, 55 in the other), and the room proportions and building proportions designed by architects. Classical musical composers have used it as an inspiration, with Bartók's Dance Suite believed to be connected to the sequence. In contemporary music Brian Transeau (aka BT) has a track in his album *This Binary Universe* called 1.618 as a salute to the ultimate ratio of the Fibonacci numbers, a number we shall discuss a little later.

Origins The Fibonacci sequence occurred in the *Liber Abaci* published by Leonardo of Pisa (Fibonacci) in 1202, but these numbers were probably known in India before that. Fibonacci posed the following problem of rabbit generation:

timeline

AD1202
Leonardo of Pisa publishes the *Liber Abaci* and Fibonacci numbers

1724
Daniel Bernoulli expresses the numbers of the Fibonacci sequence in terms of the golden ratio

Mature rabbit pairs generate young rabbit pairs each month. At the beginning of the year there is one young rabbit pair. By the end of the first month they will have matured, by the end of the second month the mature pair is still there and they will have generated a young rabbit pair. The process of maturing and generation continues. Miraculously none of the rabbit pairs die.

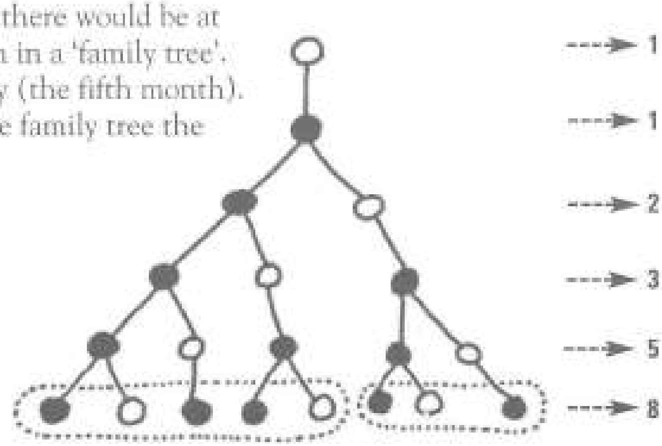
Fibonacci wanted to know how many rabbit pairs there would be at the end of the year. The generations can be shown in a 'family tree'. Let's look at the number of pairs at the end of May (the fifth month). We see the number of pairs is 8. In this layer of the family tree the left-hand group

● ○ ● ● ○

is a duplicate of the whole row above, and the right-hand group

● ○ ●

is a duplicate of the row above that. This shows that the birth of rabbit pairs follows the basic Fibonacci equation:



The rabbit population

$$\text{number after } n \text{ months} = \text{number after } (n-1) \text{ month} + \text{number after } (n-2) \text{ months}$$

Properties Let's see what happens if we add the terms of the sequence:

1 + 1 = 2
1 + 1 + 2 = 4
1 + 1 + 2 + 3 = 7
1 + 1 + 2 + 3 + 5 = 12
1 + 1 + 2 + 3 + 5 + 8 = 20
1 + 1 + 2 + 3 + 5 + 8 +13 = 33
. . .

The result of each of these sums will form a sequence as well, which we can place under the original sequence, but shifted along:

1923
Bartók composes his 'Dance Suite', believed to be inspired by the Fibonacci numbers

1963
The *Fibonacci Quarterly*, a journal devoted to the number theory of the Fibonacci sequence, is founded

2007
Sculptor Peter Randall-Page creates the 70 tonne sculpture 'Seed' based on the Fibonacci sequence for the Eden Project in Cornwall, UK

Fibonacci	1 1 2 3 5 8 13 21 34 55 89 . . .
Addition	2 4 7 12 20 33 54 88 . . .

The addition of *n* terms of the Fibonacci sequence turns out to be 1 less than the next but one Fibonacci number. If you want to know the answer to the addition of 1 + 1 + 2 + . . . + 987, you just subtract 1 from 2584 to get 2583. If the numbers are added alternately by missing out terms, such as 1 + 2 + 5 + 13 + 34, we get the answer 55, itself a Fibonacci number. If the other alternation is taken, such as 1 + 3 + 8 + 21 + 55, the answer is 88 which is a Fibonacci number less 1.

The squares of the Fibonacci sequence numbers are also interesting. We get a new sequence by multiplying each Fibonacci number by itself and adding them.

Fibonacci	1	1	2	3	5	8	<u>13</u>	<u>21</u>	34	55 . . .
Squares	1	1	4	9	25	64	169	441	1156	3025 . . .
Addition of squares	1	2	6	15	40	104	<u>273</u>	714	1870	4895 . . .

In this case, adding up all the squares up to the *n*th member is the same as multiplying the *n*th member of the original Fibonacci sequence by the next one to this. For example,

$$1 + 1 + 4 + 9 + 25 + 64 + 169 = 273 = 13 \times 21$$

Fibonacci numbers also occur when you don't expect them. Let's imagine we have a purse containing a mix of £1 and £2 coins. What if we want to count the number of ways the coins can be taken from the purse to make up a particular amount expressed in pounds. In this problem the order of actions is important. The value of £4, as we draw the coins out of the purse, can be any of the following ways, 1 + 1 + 1 + 1; 2 + 1 +1; 1 + 2 + 1; 1 + 1 + 2; and 2 + 2. There are 5 ways in all – and this corresponds to the fifth Fibonacci number. If you take out £20 there are 6,765 ways of taking the £1 and £2 coins out, corresponding to the 21st Fibonacci number! This shows the power of simple mathematical ideas.

The golden ratio If we look at the ratio of terms formed from the Fibonacci sequence by dividing a term by its preceding term we find out another remarkable property of the Fibonacci numbers. Let's do it for a few terms 1, 1, 2, 3, 5, 8, 13, 21, 34, 55.

1/1	2/1	3/2	5/3	8/5	13/8	21/13	34/21	55/34
1.000	2.000	1.500	1.333	1.600	1.625	1.615	1.619	1.617

Pretty soon the ratios approach a value known as the golden ratio, a famous number in mathematics, designated by the Greek letter ϕ. It takes its place amongst the top mathematical constants like π and e, and has the exact value

$$\phi = \frac{1+\sqrt{5}}{2}$$

and this can be approximated to the decimal 1.618033988… With a little more work we can show that each Fibonacci number can be written in terms of ϕ.

Despite the wealth of knowledge known about the Fibonacci sequence, there are still many questions left to answer. The first few prime numbers in the Fibonacci sequence are 2, 3, 5, 13, 89, 233, 1597 – but we don't know if there are infinitely many primes in the Fibonacci sequence.

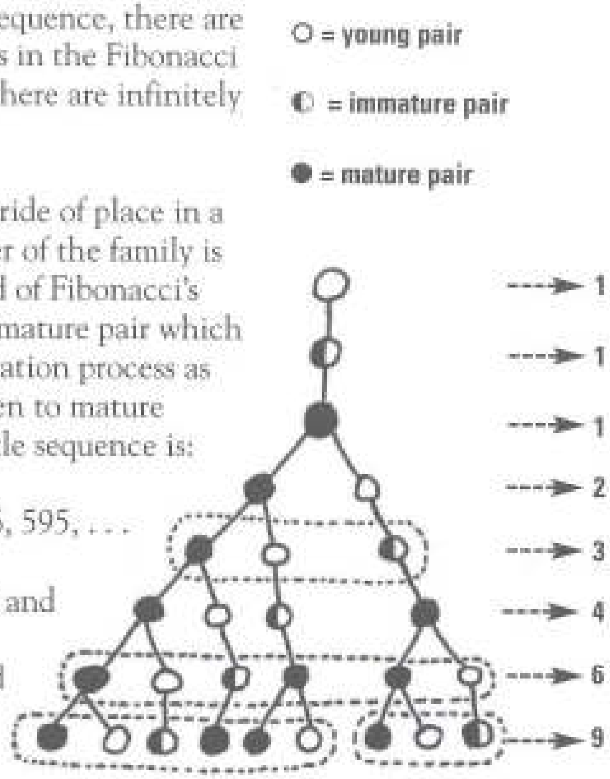
Family resemblances The Fibonacci sequence holds pride of place in a wide ranging family of similar sequences. A spectacular member of the family is one we may associate with a cattle population problem. Instead of Fibonacci's rabbit pairs which transform in one month from young pair to mature pair which then start breeding, there is an intermediate stage in the maturation process as cattle pairs progress from young pairs to immature pairs and then to mature pairs. It is only the mature pairs which can reproduce. The cattle sequence is:

1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, …

Thus the generation skips a value so for example, 41 = 28 + 13 and 60 = 41 + 19. This sequence has similar properties to the Fibonacci sequence. For the cattle sequence the ratios obtained by dividing a term by its preceding term approach the limit denoted by the Greek letter psi, written ψ, where

$$\psi = 1.46557123187676802665\ldots$$

This is known as the 'supergolden ratio'.



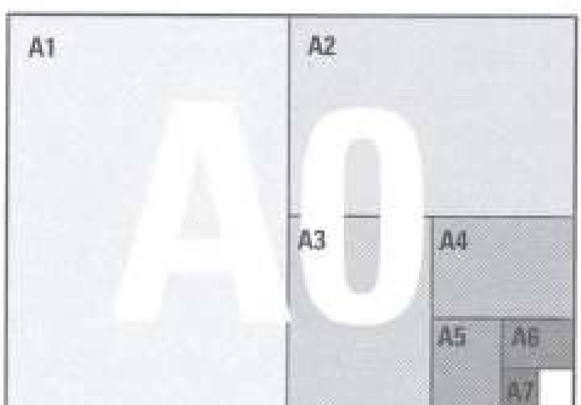
The cattle population

the condensed idea
The Da Vinci Code unscrambled

12 Golden rectangles

Rectangles are all around us – buildings, photographs, windows, doors, even this book. Rectangles are present within the artists' community – Piet Mondrian, Ben Nicholson and others, who progressed to abstraction, all used one sort or another. So which is the most beautiful of all? Is it a long thin 'Giacometti rectangle' or one that is almost a square? Or is it a rectangle in between these extremes?

Does the question even make sense? Some think so, and believe particular rectangles are more 'ideal' than others. Of these, perhaps the golden rectangle has found greatest favour. Amongst all the rectangles one could choose for their different proportions – for that is what it comes down to – the golden rectangle is a very special one which has inspired artists, architects and mathematicians. Let's look at some other rectangles first.



Mathematical paper If we take a piece of A4 paper, whose dimensions are a short side of 210 mm and a long side of 297 mm, the length-to-width ratio will be 297/210 which is approximately 1.4142. For any international A-size paper with short side equal to b, the longer side will always be $1.4142 \times b$. So for A4, $b = 210$ mm, while for A5, $b = 148$ mm. The A-formulae system used for paper sizes has a highly desirable property, one that does not occur for arbitrary paper sizes. If an A-size piece of paper is folded about the middle, the two smaller rectangles formed are directly in proportion to the larger rectangle. They are two smaller versions of the *same* rectangle.

timeline

c.300BC
The extreme and mean ratio is published in Euclid's *Elements*

AD1202
Leonardo of Pisa publishes *Liber Abaci*

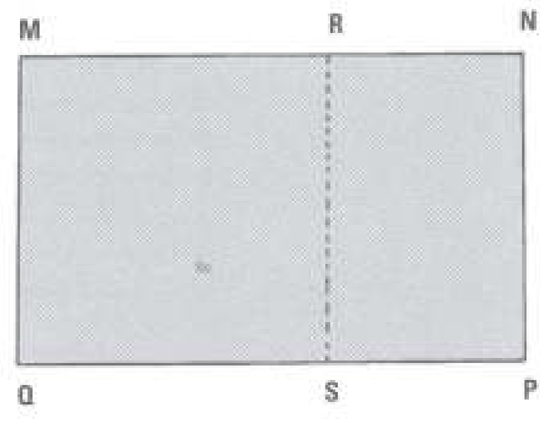
In this way, a piece of A4 folded into two pieces generates two pieces of A5. Similarly a piece of A5-size paper generates two pieces of A6. In the other direction, a sheet of A3 paper is made up of two pieces of A4. The smaller the number on the A-size the larger the piece of paper. How did we know that the particular number 1.4142 would do the trick? Let's fold a rectangle, but this time let's make it one where we don't know the length of its longer side. If we take the breadth of a rectangle to be 1 and we write the length of the longer side as x, then the length-to-width ratio is $x/1$. If we now fold the rectangle, the length-to-width ratio of the smaller rectangle is $1/\tfrac{1}{2}x$, which is the same as $2/x$. The point of A sizes is that our two ratios must stand for the same proportion, so we get an equation $x/1 = 2/x$ or $x^2 = 2$. The true value of x is therefore $\sqrt{2}$ which is approximately by 1.4142.

Mathematical gold The golden rectangle is different, but only *slightly* different. This time the rectangle is folded along the line *RS* in the diagram so that the points *MRSQ* make up the corners of a *square*.

The key property of the golden rectangle is that the rectangle left over, *RNPS*, is proportional to the large rectangle – what is left over should be a mini-replica of the large rectangle.

As before, we'll say the breadth $MQ = MR$ of the large rectangle is 1 unit of length while we'll write the length of the longer side *MN* as x. The length-to-width ratio is again $x/1$. This time the breadth of the smaller rectangle *RNPS* is $MN - MR$, which is $x - 1$ so the length-to-width ratio of this rectangle is $1/(x - 1)$. By equating them, we get the equation

$$\frac{x}{1} = \frac{1}{x-1}$$



which can be multiplied out to give $x^2 = x + 1$. An approximate solution is 1.618. We can easily check this. If you type 1.618 into a calculator and multiply it by itself you get 2.618 which is the same as $x + 1 = 2.618$. This number is the famous golden ratio and is designated by the Greek letter phi, ϕ. Its definition and approximation is given by

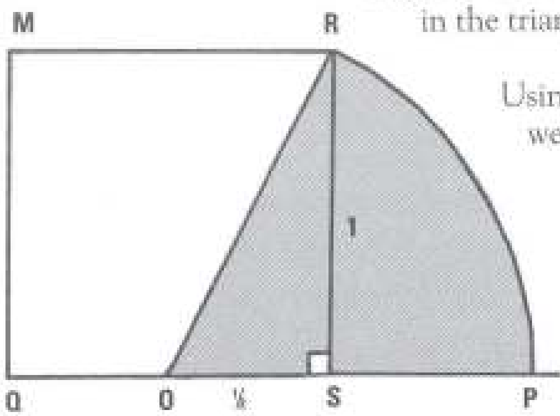
$$\phi = \frac{1+\sqrt{5}}{2} = 1.61803398874989484820\ldots$$

1509 Paciola publishes *The Divine Proportion*

1876 Fechner writes on psychological experiments to determine the proportions of the most 'aesthetic' rectangle

1975 The International Organization for Standardization (ISO) defines the A paper size and this number is related to the Fibonacci sequence and the rabbit problem (see page 44).

Going for gold Now let's see if we can build a golden rectangle. We'll begin with our square *MQSR* with sides equal to 1 unit and mark the midpoint of *QS* as *O*. The length $OS = \frac{1}{2}$, and so by Pythagoras's theorem (see page 84) in the triangle *ORS*, $OR = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}$



Using a pair of compasses centred on *O*, we can draw the arc *RP* and we'll find that $OP = OR = \frac{\sqrt{5}}{2}$. So we end up with

$$QP = \frac{1}{2} + \frac{\sqrt{5}}{2} = \phi$$

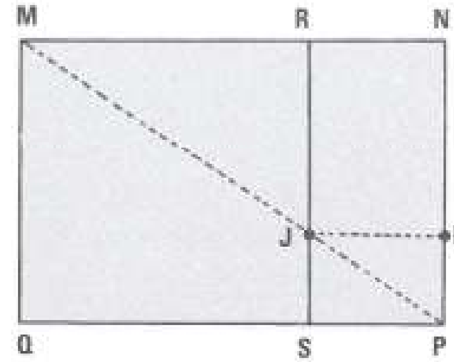
which is what we wanted: the 'golden section' or the side of the golden rectangle.

History Much is claimed of the golden ratio ϕ. Once its appealing mathematical properties are realized it is possible to see it in unexpected places, even in places where it is not. More than this is the danger of claiming the golden ratio was there before the artefact – that musicians, architects and artists had it in mind at the point of creation. This foible is termed 'golden numberism'. The progress from numbers to general statements without other evidence is a dangerous argument to make.

Take the Parthenon in Athens. At its time of construction the golden ratio was certainly known but this does not mean that the Parthenon was based on it. Sure, in the front view of the Parthenon the ratio of the width to the height (including the triangular pediment) is 1.74 which is close to 1.618, but is it close enough to claim the golden ratio as a motivation? Some argue that the pediment should be left out of the calculation, and if this is done, the width-to-height ratio is actually the whole number 3.

In his 1509 book *De divina proportione*, Luca Pacioli 'discovered' connections between characteristics of God and properties of the proportion determined by ϕ. He christened it the 'divine proportion'. Pacioli was a Franciscan monk who wrote influential books on mathematics. By some he is regarded as the 'father of accounting' because he popularized the double-entry method of accounting used by Venetian merchants. His other claim to fame is that he taught mathematics to Leonardo da Vinci. In the Renaissance, the golden section achieved near mystical status – the astronomer Johannes Kepler described it as a mathematical 'precious jewel'. Later, Gustav Fechner, a German experimental psychologist, made thousands of measurements of rectangular shapes (playing cards, books, windows) and found the most commonly occurring ratio of their sides was close to ϕ.

Le Corbusier was fascinated by the rectangle as a central element in architectural design and by the golden rectangle in particular. He placed great emphasis on harmony and order and found this in mathematics. He saw architecture through the eyes of a mathematician. One of his planks was the 'modulator' system, a theory of proportions. In effect this was a way of generating streams of golden rectangles, shapes he used in his designs. Le Corbusier was inspired by Leonardo da Vinci who, in turn, had taken careful notes on the Roman architect Vitruvius, who set store by the proportions found in the human figure.



Other shapes There is also a 'supergolden rectangle' whose construction has similarities with the way the golden rectangle is constructed.

This is how we build the supergolden rectangle *MQPN*. As before, *MQSR* is a square whose side is of length 1. Join the diagonal *MP* and mark the intersection on *RS* as the point *J*. Then make a line *JK* that's parallel to *RN* with *K* on *NP*. We'll say the length *RJ* is y and the length *MN* is x. For any rectangle, $RJ/MR = NP/MN$ (because triangles *MRJ* and *MNP* are similar), so $y/1 = 1/x$ which means $x \times y = 1$ and we say x and y are each other's 'reciprocal'. We get the supergolden rectangle by making the rectangle *RJKN* proportional to the original rectangle *MQPN*, that is $y/(x - 1) = x/1$. Using the fact that $xy = 1$, we can conclude that the length of the supergolden rectangle x is found by solving the 'cubic' equation $x^3 = x^2 + 1$, which is clearly similar to the equation $x^2 = x + 1$ (the equation that determines the golden rectangle). The cubic equation has one positive real solution ψ (replacing x with the more standard symbol ψ) whose value is

$$\psi = 1.46557123187676802665\ldots$$

the number associated with the cattle sequence (see page 47). Whereas the golden rectangle can be constructed by a straight edge and a pair of compasses, the supergolden rectangle cannot be made this way.

the condensed idea
Divine proportions