A Sketch of the Cultural Career of Mathematics

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For most of Western cultural history, mathematics enjoyed a unique "image", and a consequent prestige. Those perceptions were shaped, for centuries to come, by the achievement and outlook of ancient Greece, which saw in mathematics both a particular insight into the substance of "reality" and an unparalleled certainty of reasoning and conclusions. In the 16th and 17th centuries the founders of modern science fused with this classic legacy the further conviction—denied by the Greeks—of the enormous potential of mathematics for the description and control of the sensible world of our everyday experience. The spectacular success of the ensuing "Scientific Revolution" sent the cultural status of mathematics to unprecedented heights, whence its precision and its methodology offered inspiration and example to such diverse spheres as politics, ethics, philosophy and the arts. In the last two centuries this pre-eminence of mathematics has been threatened by internal developments—non-Euclidean geometry, Gödel's "Incompleteness" Theorem—that have cast grave doubt on its claims to unshakable sureness and absolute truth. But these same two centuries have also opened to the ancient science new and exciting vistas, which in our time embody its humanistic values, its potential for cultural enrichment, as never before. Such is the story which, in brief and superficial compass, I propose to tell.

I

The primary source and symbol of mathematics's long influence was Euclid's *Elements* (c. 300 B.C.). Later centuries found in this book a paradigm of the acquisition and organization of a body of knowledge—the foundations explicit and clear, the sequence of theorems unfolding with inexorable logic, the whole brought together in a masterpiece of arrangement and exposition. These geometrical propositions carried, of course, the stamp of an absolute certainty. But this—it may go without saying—was far from being the consequence merely of their logical form, the vacuous irrefutability possessed by statements (for example, any of the form "A or not A") that are eternally true but void of all significance. Euclid's theorems, in contrast, had for the Greeks certainty *and* content: a proposition like "the angle sum in a triangle is 180°" was held *both* to be logically necessary—in the sense that to deny it would be to end in contradiction—*and* to state a fact about the world. Of course the truth of such theorems hinged on the truth of the underlying postulates, but these—with one nagging exception—were *manifestly* true, the self-evident products of innate intuition. The one source of unease was the "parallel" postulate, which asserted (in effect) that through a point P not on a line l there passes exactly one line parallel to l; this, though plausible enough, seemed much less obviously true than the other axioms. But this minor flaw (if such it was) proved scant deterrent to Euclid's admirers. His system's astonishing double strength, its attainment of genuine knowledge with absolute sureness, dazzled the centuries. The ensuing chorus of tribute, the many attempts at emulation, would fill an anthology of diverse voices. Even thinkers not generally enamored of mathematics sometimes paid their respects; thus—to take two especially striking cases—Montesquieu, whose *Persian Letters* (1720) smile genially at the sometime excesses of *l'esprit geométrique*, confided twice to his notebooks that he took nothing as certain save the pages of Euclid,[1] while Goethe, eager advocate of a holistic philosophy of nature as against the linear thinking of geometry, nevertheless urged that all scientists strive for the great Alexandrian's clarity and rigor of argument.[2] Such examples could, of course, be much multiplied. Except only the Bible, no book in the Western heritage has had an impact so lasting and so wide.

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A second aspect of the Greek mathematical bequest ran deeper. Much ancient thought assumed, and transmitted to posterity, a kind of epistemological optimism, a confidence that the world presents an objective reality that human beings can know. At one level this is perhaps only the everyday assumption of the unreflecting; but Greek philosophy articulated it with self-conscious precision, as the belief in a fundamental congruence of our minds with the transcendent order of creation. "That which it is possible to think," said Parmenides (c. 475 B.C.) "is identical with that which can Be."[3] Some in antiquity saw the bridge between the two in the miracle of language—or rather in a primal, universal language that once mirrored perfectly the world in the word, a tongue whose loss was told in the story of Babel. But others looked instead to mathematics as the key to—because a reflection of—the ultimate cosmic order. Why mathematics? Perhaps the wide consensus enjoyed by its first principles, and the logical necessity of its conclusions, hinted at truths independent of the fleeting and fallible perceptions of individuals. But more: mathematics could be seen to reach insights into the form, the structure, the relations of things, rather than into their physical matter, and hence to grasp the enduring amid the perishable, the essential as against the contingent. Was this not the obvious lesson in the familiar distinction between (say) the triangle drawn by the geometer in the sand as an aid to his reasoning and the mental triangle that the reasoning truly contemplates? Is not the number 2 more lasting, more universal, in a word more "real", than any mortal married couple, any material pair of shoes? The tradition born of such speculation passed down the centuries.

The first explicit claim of mathematics's privileged ontological status was the mystic utterance of the Pythagoreans (6th century B.C.) that all things are numbers. This surely stemmed, at least in part, from an observation long familiar to musicians, that harmonious chords are produced by the vibration of strings whose lengths are in whole-number ratios. Its lasting impact, fittingly, would be centuries of belief, in classical and then in Christian culture, that such ratios are our minds' intimations of harmonies built deep in the nature of things, governing alike the cosmos, the state, and the relationships of individuals.[4] The assigning of true "reality" and significance to mathematical entities was given a new and historic direction by Plato. He it was who brought into philosophy, and endowed with classic statement and lasting importance, that suggestive contrast (cited above) between the geometer's idealized mental triangle and its crude, temporary physical incarnations. Mathematical objects became for Plato prime examples of the "Ideas" or "Forms" at the core of his theories of existence and of knowledge: entities apprehended by human minds but (according to him) persisting independently, and mirrored dimly by the objects our senses perceive. The famous educational program set out in the *Republic* prescribes long immersion of the student in mathematics, whose objects Plato ranked as only just below—and as offering an essential path toward—the Idea of the Good, the summit of all being and morality alike.[5]

Of course, not all contemporary opinion followed Plato in this lofty vision of mathematics. His pupil Aristotle—his only rival at the pinnacle of Greek philosophy—located true being in the physical, and held that Plato's Ideas, including the objects of mathematics, are merely human concepts, abstracted from experience but having no existence outside our minds. Yet thinkers on both sides of this divide shared crucial common ground. All could agree that mathematical objects, whatever their "reality", come first into our consciousness through the impressions of our senses. But also—and here lies a deep paradox—the Greeks were

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equally unanimous in holding that mathematics, despite these physical roots, is no tool for describing or manipulating the changing panorama of everyday experience. For their mathematics portrayed, after all, the static and the ideal, a mode of being removed and, so to say, insulated from the mutability of all things physical—perhaps by independent existence (as for Plato), perhaps by an act of abstraction (as for Aristotle). Why (the Greeks might have asked) should the necessary, permanent relations among mathematical entities be *expected* to apply to a sensible world whose only constancy was change? Plato and Aristotle "agreed that mathematics and physics do not fit, and differed only over which was at fault"[6]—the one looking down on the physical as existentially inferior, the other seeing mathematics as barred from the analysis of change by the very fixity of its abstractions. Archimedes, much the most modern mind in Greek mathematics, came close in spirit to bridging the gap, but ended by reaching only results governing the static, like the quasi-geometrical "law" of the lever. Nor was astronomy's mathematical modeling of celestial movements, by Hipparchus, Ptolemy and others, an exception; for—apart from the fact that these epicycles and eccentrics were in many minds mere mathematical fictions, designed only to "save" the phenomena—the heavens were by universal agreement precisely the region where no secular change (as opposed to the eternal, uniform whirling of celestial bodies) ever occurs.

This inability to cope with change was, in easy hindsight, one profound limitation of the cultural role of Greek mathematics. There was another. One may well surmise that in the days of Euclid and Archimedes a science which unveiled apparently immutable truth about the ultimate realities was challenge and exhilaration enough; but however that may be, this role of describing the existentially given marked also the *boundary* of Greek mathematics. The geometers dealt of course in abstractions from the sensible world, but they went beyond experience in no other way. To the Greeks, wrote Carl Boyer, "mathematics, instead of being the science of possible relations, was ... the study of situations thought to subsist in nature."[7] Greek mathematics thus declined always to invent, to define new concepts lacking physical reference, to fashion new worlds from pure imagining. To Euclid the product of two line segments was an area, the product of three was a volume, the product of four was—meaningless. In the Pythagorean discovery of incommensurability[A] the real drama, the real significance was that here the mathematical mind searched the *given* stock of numbers for the measure of (say) the diagonal of a unit square, found none, and accepted the finding—leaving to later generations the creation of irrational numbers. The genius even of Archimedes, greatest of the Greek mathematicians, shone forth in sheer technical mastery and in a marvelous methodological ingenuity, but remained bound to the objects of his experience; even Archimedes imagined no new realms.

II

The Greek mathematical achievement, only partly known and poorly understood during the Middle Ages, was fully recovered by the end of the Renaissance. That inheritance included the twin sources here suggested of the subject's perceived uniqueness, the supreme certainty claimed by geometry and the Pythagorean-Platonic vision of ultimate reality as embodied in mathematical objects and relations. From one point of view the ensuing "Scientific Revolution" consisted precisely of the union of these two Greek convictions with the radical overturning of a third, in the insistence by the "moderns" that an abstract mathematics can lead to understanding and control of the physical world. The origins of that historic reversal resist easy analysis; many would contrast the leisured, aristocratic Greek philosopher contemplating a static realm of eternal Ideas with the urgent, Faustian drive of modern Europe for mastery of nature, and this hackneyed dichotomy remains deeply suggestive. In any case practical problems that never challenged antiquity now spurred the cleverness of mathematicians: the path of a light ray refracted by a lens, the instantaneous velocity and maximum range of a projectile fired from a cannon. The resulting development wrought a strange alteration in the way some mathematical entities themselves were conceived; an ellipse, for example, which for the Greeks had been the result of the *completed* slicing of a cone by a plane, was now seen as the path traced—without completion, in a kind of "timeless time"[8]—by a moving point. Mathematics, hitherto the science of the static, was preparing to describe a world of change.

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Of course, the vital instrument of this mathematizing of nature was and remains the calculus. But the great archetypal figure of the Scientific revolution did his work independently of—indeed, largely before—the full development of that powerful algorithmic machinery. Galileo represented, with a vividness probably unparalleled before or since, the several intellectual stances that together defined in his time the unique cultural influence of mathematics. The sureness of geometrical reasoning, the ontological primacy of mathematical objects and relations, the applicability of mathematics to the physical world—all of these were for him central convictions passionately urged. The "grand book" of the universe, he declared, "is written in the language of mathematics, and its characters are triangles, circles and other geometric figures without which it is humanly impossible to understand a single word of it."[9] The final statement of his physics (*Two New Sciences*, 1638) begins its celebrated discussion of free fall with definitions and axioms rendered indubitable (so Galileo believed) by experiments, then deduces a long sequence of further results *by pure geometry;* the inquiry thus mimics exactly its Euclidean and Archimedean models, and carried for its author the same conviction. Mathematical deductions from sure premises, he felt, made the conclusions so unassailable that no experimental verification was necessary, unless to win over the obtuse.[10] More dramatically still, the truths he had reached seemed to him so absolute that no other *kind* of knowledge or explanation of the phenomena under study seemed either possible or necessary; in this sense, said Galileo, man's understanding, where mathematics can be brought to bear, rises to the level even of God's.[11] An alarmed Church duly included this breathtaking assertion of human pride among the eight counts of heresy that brought the rash Florentine to his famous trial.[12]

While Galileo's physical treatises retained the form and techniques of Greek geometry, there unfolded around him a time of rapid and revolutionary progress in pure mathematics. Pierre de Fermat touched so many aspects of this advance as almost to seem its central protagonist: founder of modern number theory, successor to François Viète in the development of symbolic algebra, co-creator with René Descartes of analytic geometry, co-founder with Blaise Pascal of the theory of probability, important contributor to the early history of the calculus. It was, of course, Isaac Newton and Gottfried Leibniz who brought this last evolution to its first great synthesis, opening the way to a century and more of truly explosive elaboration and to a corresponding surge of mathematical physics. Together, this growth of pure mathematics and (even more) the attendant examples of its triumphant application to nature, gave definitive shape to a revolution already stirring in European thought and sensibility. Here, too, Newton marks a milestone, with his *Principia* (1687); his work, though forbiddingly difficult, was better calculated than Galileo's to inspire

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the lay imagination, for even the mathematically unlettered could grasp and marvel at the reduction of universal gravitation to a simple formula describing equally the fall of an apple from a tree and the motions of the stars in their courses. So Newton became the symbol of—and eventually gave his name to—the complex of diverse reactions which together form the high-water mark in all of history for the cultural influence of mathematics.

That story has been much told,[13] but a quick survey may be forgiveable. Some aspects of this "Newtonianism" predated the *Principia,* and must be ascribed instead to the earlier progress in mathematized science, to the perennial appeal of Euclid, and to Descartes's influential urging[14] of mathematics as the key to all philosophical methodology. Baruch Spinoza's *Ethics* (1660s) remains the most Euclidean treatise ever written on a "humanistic" subject, a resolute deduction of theorem after theorem from initial definitions and axioms. The rationale is profoundly characteristic: he would discuss people's actions and appetites, said Spinoza, as if they were points or lines, for they belong to the single, uniform order of nature, and must be studied by the methods applicable in other enquiries.[15] In England Thomas Hobbes laid it down (1650s) that "Geometry is the only science which it hath pleased God to bestow on mankind," urged universal use of its techniques, and set himself the goal of inferring the immutable laws of civil society from appropriate postulates.[16] Bernard de Fontenelle (1657-1757), the pioneering popularizer of science who arguably ranks second only to Voltaire in influence on the French Enlightenment, simultaneously proclaimed the goal and hailed its realization:

> Works in ethics, politics, criticism, even eloquence, all things otherwise being equal, will be better if they bear the mark of the geometer. The order, clarity, precision and exactitude that have been prevalent in good books for some time could very well have had their source in this geometric spirit, which is being more widely spread than ever...[17]

System-builders in many spheres sought axioms that might rival geometry's in sureness; the American Declaration's "We hold these truths to be self-evident" thus breathes the spirit of its age. Arbiters of literary taste championed a mathematical plainness of discourse, whose adoption promised (in their view) to end the verbal ambiguities, the semantic fogs, with which the muddled or unscrupulous contrive to veil the face of truth. Others pursued the goal of clarity in the precision of numbers, as when Immanuel Kant decreed that no branch of learning is truly rigorous until quantified.[18] Others again saw in the new mathematics of probability the key to a "calculus" of moral and political behavior, that would bring civilized agreement to these chronically contentious realms. Indeed we should remember—or risk crucially distorting our picture of the age—that for many this espousal of mathematics and science had a fervent emotional side, the hope and conviction that the use of rational strategies, and the spread of "enlightenment" through education[B], would topple the oppressive institutions (the monarchy, the feudal order, the Church) that throve on the ignorance of the people, promoting in their stead the triumph of reason in social affairs and the indefinite perfectibility of mankind.

So strong a movement inevitably bred a backlash, as if in illustration of Newton's Third Law. Against an excessive rationalism that would make mathematical exactness and logical demonstration the sole criteria of value, dissenters proclaimed the rights of the heart over the mind, of feeling and instinct over reason, of impulse and passion over calculation. At the very dawn of the Age of Reason, Pascal—most telling of critics, by virtue of his own great mathematical gifts—set the geometric and "intuitive" minds in vivid contrast, underscoring the strengths and limitations of each.[20] As time passed such voices grew in number and vehemence, until, by the beginning of the nineteenth century, mathematics had become a prime target—and Newton the particular *bête noire*—of the Romantic protest against all that the Age of Reason stood for. This familiar indictment pictured mathematics and mathematized science as cold, dessicating abstractions that rob nature and life of all beauty, poetry and joy. John Keats phrased it in lines so lovely as almost to persuade. "Philosophy," he wrote, meaning the science of his day,

> will clip an Angel's wings,
> Conquer all mysteries by rule and line,
> Empty the haunted air, and gnomed mine,
> Unweave a rainbow.[21]

"God forbid," said William Blake, more starkly, "that Truth should be Confined to Mathematical Demonstration."[22] But if this kind of thing was the commonest of contemporary reactions there was another variety of protest, less passionate but possibly more compelling. Its most original statement came (1725) near the very height, but far from the geographic center, of the French Enlightenment, from the lonely figure of Giambattista Vico, an obscure professor of rhetoric at Naples. Vico drew on a tradition reaching back to the Middle Ages, which held that only he who makes something can truly know and understand it; on this view, for example, only God can have perfect insight into the physical world, human science (pace Galileo) reaching only intimations. Similarly—said Vico—the supposed certainty that we attain in and through mathematics merely reflects the fact that this science is our own creation; its deepest theorems are only the spinning out of our own assumptions.[23] Vico was echoed (unknowingly) in mid-century by Georges Louis Leclerc, Comte de Buffon. Doubtless this great writer on natural history felt deeply the sense, shared by many biologists[C], that the tidy categories of mathematics can never do justice to the teeming variety and vast complexity of living things. In any case he held that mathematics begins with abstractions and remains confined to them, for "there is nothing in this science that we have not put there and the truths that we draw from it can only be different expressions of the same suppositions that we have employed."[25]

III

In its time this was very much a minority view. Far more commonly, mathematics was still seen in, say, 1800 A.D., as the Greeks had seen it, as a science grounded in self-evident perceptions of the physical world, attaining a certainty beyond the merely tautological, and allowing—though confined to—an exact description of an objective reality. But now an epochal change was on the horizon, the result not of external criticism but of mathematics's own inherent progress. The great watershed was the creation of non-Euclidean geometry. After hesitant beginnings in other hands, mature formulations were reached independently by Carl Friedrich Gauss in Germany (from 1792), Wolfgang and Johann Bolyai, father and son, in Hungary (1815), and Nikolai Lobachevsky in Russia (1826). All made a single change in the axiomatic base of Euclidcean geometry, in effect replacing the troublesome "parallel" postulate by an alternative which denied it, and all concluded that the resulting geometry is mathematically just as valid, because just as consistent (free from internal contradiction) as Euclid's. To those who could and would heed, the lesson was clear. The proud claim of mathematics to absolute truth had been a delusion. Vico and Buffon had been right: the validity of any mathematical theory hinges only on its underlying assumptions. And because such assumptions are arbitrary, there is no reason to credit the resulting theories with any relevance to our understanding of nature. In particular the old idea that Euclidean geometry describes physical space, so far from being self-evident, might not even be true; the new geometry might serve as well or better, and (as Gauss already saw) the choice between the two had become empirical, to be made by such observation and measurement as might avail. The Queen of the Sciences tottered precariously on her pedestal.

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So understood, this historic watershed would indeed seem a harbinger of unrelieved gloom. But already the younger Bolyai drew from his non-Euclidean geometry a more heartening promise. "From nothing," he exulted, "I have created an entirely new world."[26] Succeeding decades would echo that perspective, and that enthusiasm, many times. No doubt the links of mathematics to the world of experience, though far from actually severed—for the 18th century's splendid development of rational mechanics seemed, after all, as valid as ever—had been rendered more mysterious and problematic than before.[27] But now beckoned, as if in compensation, a widened picture of mathematics as outstripping the *limits* of mundane reality, a creative endeavor subject always to the requirement of consistency but otherwise knowing no boundaries save those of imagination. Of course, there had been some earlier hints of this passing beyond physical experience, this leveling of the barrier that had always confined the Greeks; but full acceptance and exploitation had tended to lag behind. Thus the 16th century introduced "imaginary" entities as roots of algebraic equations, but mathematicians at first viewed these with deep unease, and they waited nearly three hundred years for recognition as legitimate numbers. Likewise the analytic geometry of Descartes and Fermat inherently invited extrapolation to dimensions beyond the third. Yet here, too, the real breakthrough came only in the early 19th century. These cases are typical: only in that age, coincident with the rise of non-Euclidean geometry, did mathematicians begin to feel and pursue a real sense of creative freedom. Various developments reflected or reinforced the trend: the growing emphasis on abstract structures as opposed to their concrete interpretation, the apparent mathematical taming by Georg Cantor (1870s) of a realm of thought palpably without physical counterpart: the infinite. There was—there still is—a darker side: notoriously, the tendency to build systems on arbitrary postulates has led in some hands to an arid and sterile formalism. But, at the other extreme, minds within and outside mathematical ranks were now stretched and beguiled by such novel fantastications as the Möbius (one-sided) strip and the Koch "snowflake" (whose infinite perimeter encloses a finite area); here were delights capable, perhaps, of disarming even the hostile Romantics. Nor did the new boldness of imagination compromise the power of mathematics in the study of nature. On the contrary—some theories born of no motive save their intrinsic mathematical interest later proved almost magically fruitful in outside

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applications: a version of non-Euclidean geometry due to Bernhard Riemann (1854) awaited Einstein's general relativity, and Arthur Cayley's matrix algebra (1855) lay to hand when, seventy years later, Werner Heisenberg needed it in the development of wave mechanics. A hundred years after the severe foundational crisis threatened by non-Euclidean geometry, mathematics seemed more vigorous than ever, in internal richness and external relevance alike.

The 20th century produced another famous result with the potential to shatter confidence. Kurt Gödel showed (1931) that *any* attempt to axiomatize even so simple a mathematical system as elementary arithmetic leaves "undecidable" propositions (that is, propositions that can be neither proved nor disproved within the system), and moreover that the consistency of such a system cannot *in principle* be proved without the use of methods whose own consistency is equally (or more) in need of justification. Taken together, these findings imply fatal limitations on any hope of deducing a body of knowledge completely and certainly by (finitary) formal methods. And yet even this apparently stunning setback has in practice proved survivable. Some working mathematicians have merely responded with a "bland indifference"[28]; but others have urged, even as the Greeks urged long ago, that intuition guarantees both the objective validity and the logical coherence of the foundations. Gödel himself wrote:

> Despite their remoteness from sense experience, we do have something like a perception of the objects of set theory, as is seen from the fact that axioms force themselves on us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception... They, too, may represent an aspect of objective reality.[29]

Perhaps there is in such pronouncements an almost Kierkegaardian leap of faith—another aspect of modern mathematics that would have appealed to the Romantics. In any case it seems that, despite all undermining of foundations, most researchers still ascribe to their theorems a timeless, unconditioned truth; Euclid may be dethroned, but Plato survives. Nor, manifestly, do foundational qualms slow the attempted use of mathematics in other spheres. Economics, psychology, sociology, anthropology, history, biology, medicine—all constantly seek in numbers and formulas the precision long enjoyed by the physical sciences. These, in turn, continue to draw applications from even the most abstract or exotic of mathematical creations; so infinite-dimensional spaces play a vital role in modern physics, and the captivating new study of "fractals" offers a tool for the fruitful modeling of mountains and coastlines. At deeper levels, many of our most central formulations of scientific inquiry—quantum theory, relativity theory, cosmology—have come to rely on mathematics as the language of nature in a sense transcending any urged even by Galileo, as a unique, necessary, *probably untranslatable* encoding of such perception and understanding of physical reality as we may hope to attain.

But beyond all this, beyond all considerations of external reference, mathematics remains for many a self-contained world of enchantment, an inexhaustible realm of the strange, the diverting, the beautiful, the intellectually challenging. And hence comes a capacity for cultural impact and enrichment very different from, but not less than, the Age of Reason's exuberant embrace. To be sure, the subject's sheer proliferation, the technical vocabulary, the subtlety of reasoning, will always be formidable barriers to lay understanding. But much nontrivial diffusion occurs, and rewards enormously. "Touch on even its more abstruse regions," writes George Steiner—he is thinking of a wide area comprising both pure and applied mathematics—"and a deep elegance, a quickness and merriment of the spirit come through." He gives as example the Banach-Tarski Paradox, in its common popularization to the effect that a spherical pea can be finitely divided into pieces rearrangeable by rigid motions into another sphere the size of the sun; "what surrealist fantasy yields a more precise wonder?"[30] Through such fascination the new realms explored by mathematics have helped to shape many significant works of literary and visual art. The "fourth dimension" has intrigued novelists of the stature of Dostoevsky, Conrad and Proust;[31] non-Euclidean geometry played a role in the revolutionary visions of Cubist painting.[32] The borrowing goes on: let the cunningly interwoven, subtly varied "tessellations" (repeated patterns) in Maurits Escher's famously intricate designs, and the absorbed play with post-Cantorian infinities in Jorge Luis Borges' delectable *ficciones,* stand as two especially distinguished recent examples. That such cases are not even more numerous—in particular the echoes of mathematics in modern science fiction are surprisingly faint[33]—suggests an ongoing task of dissemination for the research community and for educators. But the potential is surely very great. Stripped of its ancient certitude but still a prodigiously vital and growing enterprise, illumining the world of physical experience but no longer confined there, modern mathematics expresses the human spirit in three distinct but intertwining ways, as a "man-made universe"[34] to be cultivated and cherished for its own sake, as an indispensable instrument for our understanding of nature, and as a limitless source and playground of delighted imagination.

Most researchers still ascribe to their theorems a timeless, unconditioned truth; Euclid may be dethroned, but Plato survives.

A. That is, the discovery that there exist pairs of geometrical "magnitudes"—line segments, or areas, or volumes—with the property that no unit magnitude of their type measures both of them an integer number of times. For us, but *not* for the Greeks, two magnitudes are incommensurable in this sense if and only if the ratio of their measures is an irrational number.

B. Some filtering down of the intellectuals' passion for mathematics is hinted by anecdotal evidence, as of the young ladies in late-17th-century France who allegedly refused otherwise eligible suitors for having no new ideas on the squaring of the circle.[19]

C. Aristotle's opinions on mathematics can usefully be viewed from this perspective, and Ernst Mayr is its eloquent advocate in our time.[24]

NOTES

1. Montesquieu, *Oeuvres Complètes* (Paris, 1966), pp. 856, 959; cf. *Lettres Persanes,* Letter 129.

2. H.B. Nisbet, *Goethe and the Scientific Tradition* (London, 1972), p. 50.

3. Parmenides, frag. B3, in Kathleen Freeman, *Ancilla to the Pre-Socratic Philosophers* (Cambridge, Mass., 1983), p. 42, note 2.

4. Cf. the fascinating book of Leo Spitzer, *Classical and Christian Ideas of World Harmony* (Baltimore, 1963).

5. Plato, *Republic*, Bk. VII.

6. C.C. Gillispie, *The Edge of Objectivity* (Princeton, N.J., 1960), p. 15.

7. C.B. Boyer, *The History of the Calculus and its Conceptual Development* (New York, 1959), p. 25.

8. A. Koyré, "The Significance of the Newtonian Synthesis", in his *Newtonian Studies* (Chicago, 1965), pp. 10-11.

9. Galileo, "The Assayer", in S. Drake, ed., *Discoveries and Opinions of Galileo* (Garden City, N.Y., 1957), pp. 237-38.

10. Galileo, letter to Carcavi, in *Opere*, Vol. 17, pp. 90-91.

11. Galileo, *Dialogue Concerning the Two Chief World Systems,* tr. S. Drake (Berkeley and Los Angeles, 1970), pp. 103.

12. Ibid., p. 474.

13. Notably by M. Kline, *Mathematics: A Cultural Approach* (Reading, Mass., 1962), Chaps. 20-22; idem., *Mathematics in Western Culture.*

14. E.g. Discourse on the Method, pt. 2, in J. Cottingham et al., eds., *The Philosophical Writings of Descartes,* Vol I, p. 120.