Log Functions and Exponentials

One of the most important jobs logs do is help us to model real-world phenomena, or variables. Suppose that this variable f is modelled by an exponential:

$$f(x) = \alpha e^{\beta x}$$

Note: this is equivalent to the form

$$f(x) = e^{m+bx}$$

the composition of an exponential function and an affine (aka "linear") function. We can see this by using properties of exponents[1]:

$$f(x) = e^{m+bx} = e^m e^{bx} \equiv \alpha e^{\beta x},$$

where $\alpha = e^m$ and $\beta = b$.

Now that we've got that out of the way, if we use logs to transform f by taking logs of both sides, we get

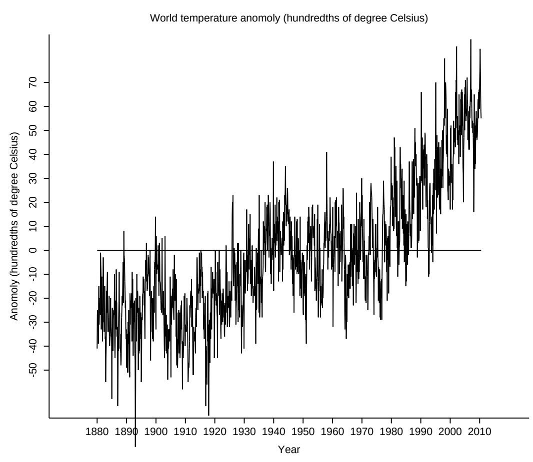
$$\ln(f(x)) = \ln(\alpha e^{\beta x})$$

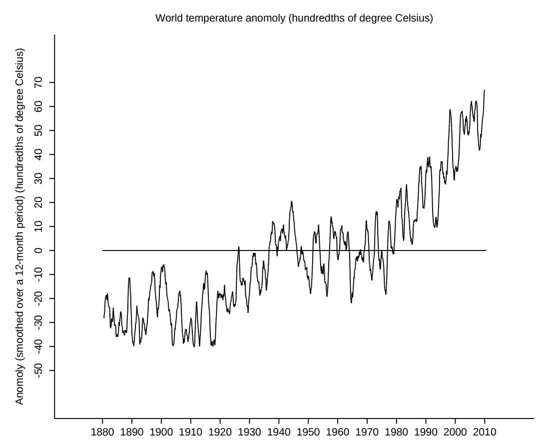
and we use properties of logs[2] to rewrite that as

$$\ln(f(x)) = \ln(\alpha) + \ln(e^{\beta x}) = \ln(\alpha) + \beta x$$

At right we have a linear function ($\ln(\alpha)$ is just an ugly constant): at left is the log of f. We don't actually know f, but we have collected some data for f, so if we log-transform that data, it should appear linear. If it does, then we might have been right about the variable f – maybe an exponential model is the appropriate one.

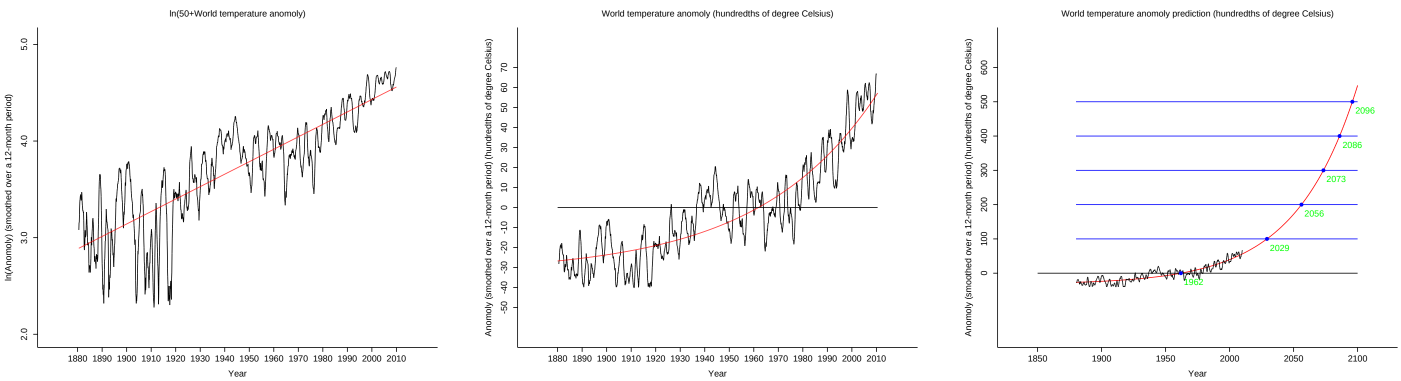
Figure 1: Source: `http://data.giss.nasa.gov/gistemp/tabledata/GLB.Ts+dSST.txt`. "Best estimate for absolute global mean for 1951-1980 is 14.0 deg-C or 57.2 deg-F, so add that to the temperature change if you want to use an absolute scale." At left, the original time-series; at right is smoothed data (averaged over 12 months).





[1] Those are so important! Please memorize them!
[2] Memorize those too!

Figure 2: log-transformed data (left) is modelled with a linear function; at right is the original (smoothed) data, modelled using an exponential. Below all we have the predictions based on our model.



a. What do you think of the linear fit? Is it "linear enough" to assert the exponential model?

b. This from the IPCC (Intergovernmental Panel on Climate Change) summary report, 2007[3]: "There is medium confidence that approximately 20 to 30% of species assessed so far are likely to be at increased risk of extinction if increases in global average warming exceed 1.5 to 2.50C (relative to 1980-1999). As global average temperature increase exceeds about 3.50C, model projections suggest significant extinctions (40 to 70% of species assessed) around the globe."

Comment, based on our work here.

[3] `http://www.ipcc.ch/pdf/assessment-report/ar4/syr/ar4_syr_spm.pdf`

Now, consider this problem. Two points are given: (1,6) and (3,24). These points lie on an exponential graph of the function of form $f(x) = Ca^x$.

a. Find C and a directly.

b. Find C and a by first taking logs (use the data!), and fitting a straight line.