**Section Summary: Taylor and MacLaurin Series**

# 1 Definitions

**Taylor series of $f$ about $a$ (assuming $f$ has derivatives of all orders):**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The $n^{th}$**-degree Taylor polynomial of $f$ at** $a$:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Then $R_n(x) = f(x) - T_n(x)$ is the called the **remainder** of the Taylor series.

**Maclaurin series**: a Taylor series centered about $x = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

# 2 Theorems

If $f$ has a power series representation (expansion) at $a$, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

If $f(x) = T_n(x) + R_n(x)$, and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then $f$ is equal to the sum of its Taylor series on the interval $|x-a| < R$.

**Taylor's inequality**: If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for} \quad |x-a| \leq d$$

# 3 Properties, Hints, etc.

Power series can be added and subtracted just like polynomials (but be aware of possibly different intervals of convergence). While they can also be multiplied and divided like polynomials, they're quite cumbersome to manipulate this way. We're often only interested in the first few terms, however, which makes this an occasionally useful option.

Some important Maclaurin series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (-1,1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty,\infty)$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad (-\infty,\infty)$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad (-\infty,\infty)$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad [-1,1]$$

# 4 Summary

This is the crowning glory of sequences and series. This material clarifies statements like these: "$\sin(x) \approx x$ about $x = 0$." Or, equivalently,

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

This type of analysis is clear once Taylor series are understood.

We can obtain Taylor series by term-by-term differentiation: suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

with radius of convergence $R > 0$. Then $f(0) = c_0$ (the series is easy to evaluate at $x = 0$!). We've seen that we can differentiate series, so, provided $f$ is differentiable,

$$f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

Again, evaluate at 0: $f'(0) = c_1 \cdot 1 = c_1$. So $c_1 = f'(0)$.

Continuing,

$$f''(x) = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

Again, evaluate at 0: $f''(0) = c_2 \cdot 2 \cdot 1 = 2c_2$. So $c_2 = \frac{f''(0)}{2}$.

In general,

$$c_n = \frac{f^{(n)}(0)}{n!}$$

and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

This is the so-called "Maclaurin series" (a special type of Taylor series, expanded about $x = 0$).

We can also obtain Taylor series by building up: the objective is to approximate a function about an abscissa $a$ as well as possible by successively larger polynomials. If you only had a constant function to work with, you'd choose $p(x) = c = f(a)$. If you had a linear function to work with, you'd choose $p(x) = ax + b = f'(a)(x - a) + f(a)$ (check that this gets both the function value right, and the slope right). Continuing in this fashion, you'd get

$$\frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a)$$

$$\frac{f'''(a)}{3!}(x-a)^3 + \frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a)$$

etc.