Section Summary: The Comparison Tests

1 Theorems

The Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

a. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

b. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, $\sum a_n$ is also divergent.

The Limit Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

2 Summary

The sense of the comparison tests is two-fold:

a. one test says that if terms of a series are bounded below by a divergent series, then the series diverges; and similarly if terms of a series are bounded above by a convergent series, then the series converges. This is similar to other comparison theorems we have encountered (e.g. integral comparisons).

b. The other test says that if terms of two series are proportional in the limit (i.e. $a_n = cb_n$, then they converge or diverge together.

Typical candidate series for comparisons are p-series or geometric series, because we have good theorems about their convergence.

It's important to realize that these comparison test conditions only have to be met *eventually*: for issues of convergence and divergence, we don't care what happens to the first 100, or 1000, or gazillion terms: it's only what happens to the infinite tail that is really crucial.