Hyperbolic Functions

a. Definitions

The names of these functions are obviously derived from their trigonometric relatives. The reason these functions have these names is that they resemble the trig functions in terms of their relative behaviors and in terms of their identities:

- $$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

  (an odd function),

- $$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

  (an even function), and

- $$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

  (an odd function, as the quotient of an odd and an even function).

Pronounce them "sinsh", "cosh", and "tansh", however.

The graph of $\cosh(x)$ is the catenary, or the "hanging chain". All the power lines that you've been looking at over the years are hanging in the form of **catenaries** (perhaps you thought that they were parabolas, but they're not).

Unlike their trig cousins, however, *sinh* and *cosh* both blow up to infinity. Here are some of their properties and identities – look for the similarity to the trig functions:

- $$\cosh^2(x) - \sinh^2(x) = 1$$

- $$1 - \tanh^2(x) = sech^2(x)$$

- $$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

- $$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

Derivative properties:

- $$\frac{d}{dx}(\sinh(x)) = \cosh(x)$$

- $$\frac{d}{dx}(\cosh(x)) = \sinh(x)$$

- $$\frac{d}{dx}(\tanh(x)) = sech^2(x)$$

Inverses: We can find the inverses in the usual way – by solving $\sinh(x) = y$ for $x = \sinh^{-1}(y)$. We get

- $$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}) \qquad\qquad x \in \mathbb{R}$$

- $$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1}) \qquad\qquad x > 1$$

- $$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad\qquad -1 < x < 1$$

We can use straightforward derivatives of logs to get the derivatives of the inverse hyperbolic functions:

- $$\frac{d}{dx}(\sinh^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$$

- $$\frac{d}{dx}(\cosh^{-1}(x)) = \frac{1}{\sqrt{x^2-1}}$$

- $$\frac{d}{dx}(\tanh^{-1}(x)) = \frac{1}{1-x^2}$$

b. Summary

The hyperbolic functions have many properties similar to the properties of the trigonometric functions, which explains their names – the three most important being sinh, cosh, and tanh.

One of the reasons they're so important is because one of their family, cosh, has as its graph the "hanging chain". The St. Louis arch is actually in the shape of a catenary. The shape has important engineering properties.