Order of Convergence of the Secant Method

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March 26, 2015

1 From Newton to Secant

Consider $f(x)$, with root r. Assume that $\{x_k\}$ is a sequence of iterates obtained using the secant method, and converging to r.

Defining the errors $e_k = x_k - r$, we conclude that convergence of the iterates x_k to r implies that $\lim_{k \longrightarrow \infty} e_k = 0$.

The secant method is derived from Newton's method (quadratic order of convergence) by replacing a derivative by a finite difference:

$$x = g(x)$$

where

$$g(x) = x - \frac{f(x)}{f'(x)}$$

which converges quadratically with

$$\lim_{k \longrightarrow \infty} \frac{|e_{k-1}|}{|e_k|^2} = \left| \frac{f''(r)}{2f'(r)} \right|$$

Of course we use this iteratively, so that

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Instead of knowing and computing the derivative, we can use an approximation to the derivative in this iterative scheme (if we have two starting points):

$$x_{k+1} = x_k - \frac{f(x_k)}{\frac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}}} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$$

It is this scheme, the secant method, that we want to analyze for convergence.

2 Secant's Convergence is order ϕ (the golden ratio)

2.1 The Taylor Series and Big-Oh Notation

This is what we did last time, arriving at

$$\lim_{k \to \infty} \frac{e_{k+1}}{e_k e_{k-1}} = \frac{f''(r)}{2f'(r)}.$$

2.2 Computation of p

Order of Convergence is defined on page 87. We seek $p > 0$ and $C > 0$ such that

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^p} = C.$$

Assume that they exist. Define $M \equiv \dfrac{f''(r)}{2f'(r)}$.

Since

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k e_{k-1}} = M,$$

we have

$$\lim_{k\to\infty} |e_{k+1}| = \lim_{k\to\infty} |e_k e_{k-1} M|.$$

Then

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^p} = \lim_{k\to\infty} \frac{|e_k e_{k-1} M|}{|e_k|^p} = \lim_{k\to\infty} \frac{|e_k|^{1-p}}{|e_{k-1}|^{-1}} |M| = C;$$

but also

$$\lim_{k\to\infty} \frac{|e_k|^{1-p}}{|e_{k-1}|^{-1}} = \lim_{k\to\infty} \frac{|e_{k+1}|^{1-p}}{|e_k|^{-1}} = \frac{C}{|M|}.$$

So we have two limits involving e_{k+1} and e_k:

$$\lim_{k\to\infty} \frac{|e_{k+1}|^1}{|e_k|^p} = C \text{ and } \lim_{k\to\infty} \frac{|e_{k+1}|^{1-p}}{|e_k|^{-1}} = \frac{C}{|M|}.$$

Now the only way two ratios of these terms converging to 0 can both converge to finite non-zero values is for the **relative ratios of the exponents** to be the same. This means that

$$\frac{1}{p} = \frac{1-p}{-1}$$

or $(1-p)p = -1$; that is, p is a root of the quadratic

$$p^2 - p - 1 = 0$$

As one can compute, the positive root $p = \dfrac{1+\sqrt{5}}{2} \approx 1.618$ is the order of convergence of the secant method.

If you're a little confused about the relative ratios argument, think about squares, for example: if

$$\lim_{k\to\infty} \frac{|e_{k+1}|^1}{|e_k|^p} = C \text{ then } \lim_{k\to\infty} \frac{|e_{k+1}|^2}{|e_k|^{2p}} = C^2.$$

Both converge to a finite, non-zero number, and the relative ratios of the exponents are equal.

Here's another concrete example, if that still didn't sit well: we know, for example, that

$$\sin(x) = x - \frac{x^3}{3!} + O(x^5)$$

Hence

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + O(x^4)$$

and so

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

This means, for example, that

$$\lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)^2 = \lim_{x\to 0} \frac{\sin(x)^2}{x^2} = 1$$

So the ratio of the exponents on the terms (1/1 and 2/2) has to be the same for convergence to a finite, non-zero value. Thus

$$\lim_{x\to 0} \frac{\sin(x)^2}{x} = 0$$

and

$$\lim_{x\to 0} \frac{\sin(x)}{x^2} = \pm\infty.$$

2.3 Computation of C

Now that we've got p, it's simple to get C: from

$$\lim_{k\to\infty} \frac{|e_{k+1}|^1}{|e_k|^p} = C \text{ and } \lim_{k\to\infty} \frac{|e_{k+1}|^{1-p}}{|e_k|^{-1}} = \frac{C}{|M|}.$$

So if I raise the quantity on the left to the power $1-p$, then the limits must be the same: i.e.

$$\lim_{k\to\infty} \left(\frac{|e_{k+1}|^1}{|e_k|^p}\right)^{1-p} = C^{1-p} = \lim_{k\to\infty} \frac{|e_{k+1}|^{1-p}}{|e_k|^{-1}} = \frac{C}{|M|}.$$

Hence

$$C^{1-p} = \frac{C}{|M|}.$$

or $C = |M|^{\frac{1}{p}} = \left|\frac{f''(r)}{2f'(r)}\right|^{p-1} \approx \left|\frac{f''(r)}{2f'(r)}\right|^{.618}.$