MAT360 Exam 2 (Spring 2015): Root-finding and Interpolation

Name:

Directions: **Each page is worth 20 points, and you must skip one page. Write "skip!" clearly on that page.** Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

Problem 1. (20 pts) Taylor series

a. Use Taylor's theorem to expand $f(a+h)$ about $x=a$ for a sufficiently differentiable (i.e. smooth) general function $f(x)$. Go as far as the cubic term, and include the $O(h^4)$ error term.

b. Now we'll use that expansion to approximate$f(x)=\cos(x)$ about the point $x=\frac{\pi}{4}$. Write out the expansion explicitly for this case.

c. Continuing on, if we use the Taylor cubic obtained in the previous step to approximate f on the interval $[0, \frac{\pi}{2}]$, provide a reasonable bound for the error we might make (based on the Taylor series). Where will it be largest? Where smallest?

Problem 2. (20 pts) Suppose that you want to use a Hermite cubic spline to approximate the function $f(x) = \cos(x)$ on the interval $[0, \frac{\pi}{2}]$. Suppose we use two Hermite cubics, $h_1(x)$ between points $x_0 = 0$ and $x_1 = \frac{\pi}{4}$, and $h_2(x)$ between $x_1 = \frac{\pi}{4}$ and $x_2 = \frac{\pi}{2}$.

a. Write down the conditions that the two Hermite cubics $h_1(x)$ and $h_2(x)$ must satisfy.

b. Write an explicit formula for the cubic $h_1(x)$. You may leave divided-differences unevaluated (if you use them).

c. Where will the errors in the spline be largest? Bound the error.

Problem 3. (40 pts) Consider the three points

x_i	$f[x_i]$
$x_0 = 0$	1
$x_1 = 2$	2
$x_2 = 4$	1

a. Complete the divided-difference table.

b. Write the Newton interpolating polynomial obtained by adding the points in the order x_2, x_1, and x_0.

c. Write the interpolating polynomial in Lagrange form.

d. Write the interpolating polynomial in the very simplest and sensible form for computation.

e. Write the interpolating polynomial expanded about $x = x_2$ using Horner's rule for evaluating polynomials.

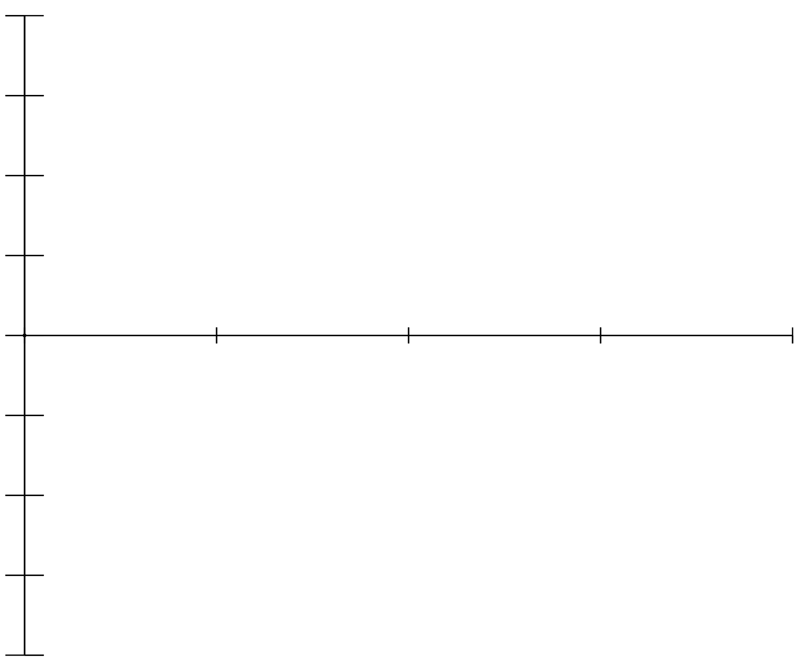
f. What is the difference between these polynomials?

g. (Problem 3, continued.) Suppose that x_0, x_1, and x_2 are the first three guesses for the (positive) root of $f(x)$.

a. What would Muller's method give as the next guess?

b. Why could x_2 **not** be the guess provided by the secant method, using x_0 and x_1?

c. Draw a sample function interpolating the points x_0, x_1, and x_2 such that x_1 and x_2 would be the succession of iterates (guesses) provided by Newton's method starting from initial guess x_0. Describe or show explicitly why your function works to produce these iterates.



Problem 4. (20 pts) Consider the root-finding problem for $f(x)$ with root $x = r$.

a. Define **Order of convergence** and **asymptotic error constant** (for a root-finding scheme).

b. Describe the calculation of the next iterate for each of these methods, and give their order of convergence (and asymptotic error constant, if known):

   a. Newton's method, given x_0:

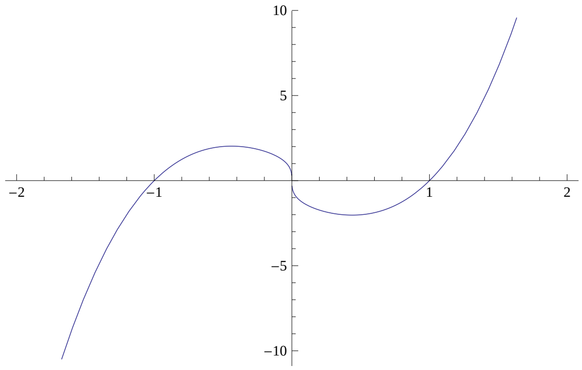
   b. The Secant method, given x_0 and x_1:

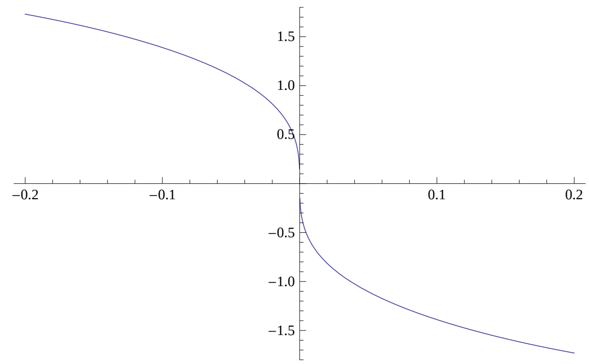
   c. A generic fixed-point iteration scheme based on $g(x)$, given x_0:

c. Why is Newton's method better than a generic fixed-point iteration scheme?

d. What is the approximate order of convergence of Muller's method?

Problem 5. (20 pts) Here are two graphs of the function $f(x) = 3\left(x^3 - x^{\frac{1}{3}}\right)$:





a. Bisection: describe all starting intervals containing more than one root that will result in bisection producing the root $x = 0$. Why does bisection have so much trouble finding that root?

b. Describe why Newton's method will never arrive at the root $x = 0$ (unless that's given as the starting guess!).

c. What must be true of successive iterates x_i and x_{i+1} so that secant will find the root $x = 0$? Is it possible to have two such iterates?