Section Summary: 2.2

a. Definitions

In our last section, we defined the **derivative of a function** f **at** a as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists. Now we generalize this definition by permitting a to vary, turning the derivative into a function of the variable x.

- **derivative of a function** f - denoted by $f'(x)$,

  $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

  Synonyms for the derivative function:

  $$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f = Df(x) = D_x f(x)$$

  This function simply associates with each element $x = a$ of the domain the slope of the curve (i.e. the slope of the tangent line) at the points $x = a$ (provided it exists).

- **differentiation operators** - The symbols D and $\frac{d}{dx}$ in the synonym tree above are called **operators**: they operate on the function f to produce a new function (the derivative function) f'.

  The difference between operators and functions is one of **domain**: an operator takes as its domain something more general than numbers. In this case, these operators take functions and return functions; their domains are sets of functions, and their ranges are functions. Thus we might write

  $$D : f \longrightarrow f'$$

  whereas an ordinary function, like $g(x) = x^2 - 3x + 1$, would be written

  $$g : x \longrightarrow x^2 - 3x + 1$$

- **differentiation** - The operation referred to above: the operation of creating the derivative function g' from a given function f.

- Here's another popular way of thinking about differentiation: the infinitesimals dx and dy are like dust left as the real, finite Δx and Δy approach 0:

  $$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$$

  This produces some synonyms for the derivative at $x = a$, $f'(a)$:

  $$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right]_{x=a}$$

  We think of all of these as the derivative function being **evaluated at** a.

- **differentiable** - A function f is **differentiable at** a if the derivative $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

  We might think of differentiability implying the ability to get a derivative at a point; it implies the existence of a derivative.

- **vertical tangent line** - a place where the tangent line to the curve is vertical. The slope in this case is infinite, so the derivative doesn't exist at points with vertical tangent lines. An example of a function with an infinite slope is the function $f(x) = x^{\frac{1}{3}}$ (check it out on your calculator).

- Higher derivatives

  – **second derivative**: the derivative of the derivative of f is called the second derivative of f, denoted $f''(x)$.
    In the context of the position function of a particle, the second derivative of the position is known as the **acceleration** of the particle.

  – **third derivative**: the derivative of the second derivative of f is called the third derivative of f, denoted $f'''(x)$.
    In the context of the position function of a particle, the third derivative of the position is known as the **jerk**.

  – n^{th} **derivative**: if we differentiate a function n times, we obtain the n^{th} derivative (which doesn't necessarily have a nice intuitive sense). It will be denoted

$$f^{(n)}(x)$$

b. Theorems

If f is differentiable at a, then f is continuous at a.

For the derivative to exist, the function must be defined at a (so $f(a)$ exists); then the limit of $f(x)$ must exist and approach the value $f(a)$ at a. Let's look at the derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The denominator heading for zero means that the numerator must, as well – or the limit will be infinite. So, for the derivative to exist, the numerator must head for zero as well:

$$\lim_{x \to a} (f(x) - f(a)) = 0$$

That is, $\lim_{x \to a} f(x) = f(a)$: hence, differentiability implies continuity.

Note: it is not true that if f is continuous at a, then f is differentiable at a. For example, a continuous function with a corner at $x = a$ is not differentiable there. An example is the absolute value function $|x|$ at $x = 0$: while continuous at 0, the function has a corner there.

c. Properties/Tricks/Hints/Etc.

- Ways in which a function can fail to be differentiable at a:
  - discontinuities of all sorts
  - corners in continuous functions
  - continuous functions at places with vertical tangent lines: the slope of such lines is either $\pm\infty$, but the derivative must be finite.

d. Summary

In this case we take the derivative as defined in section 2.1 one step further: if for each value a of the domain there is a derivative $f'(a)$, then we could write

$$x \rightarrow f'(x)$$

to mean that with every value x of the domain there is associated a value $f'(x)$. Hence we can think of f' as a **function** (the slope function, which gives the slope of the curve at any point of the graph - where defined).

The slope might not be defined for a number of reasons: the graph may be discontinuous (have a hole, or jump, or infinite discontinuity); the graph of a continuous function may have a corner (where the tangent line is not defined); or a smooth, continuous function may have a tangent line with infinite slope.

Note that in the definition of the derivative function we simply replace the value of a with x: we've been thinking of a as a fixed number, but now that we want to think of a as varying, we replace it with x (to make you think of it as a variable).

This section also introduces the concept of higher derivatives, and in particular the second and third derivatives. The second derivative is just the derivative of the derivative, so everything we've learned about derivatives applies to it. The same is true for higher derivatives, although they just get farther removed from the original function f in general.

You should pay special attention to the graphical relationship between successive derivatives. For example, $f''(x) = 0$ when the function $f'(x)$ has a horizontal slope. Knowing things about higher derivatives gives us more information about the shape of f (as the roots of the derivative function indicate points of horizonal slope on the graph of f).