Section Summary: 3.4

a. Definitions

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  $$\lim_{x\to\infty} f(x) = L$$

  means that as x gets arbitrarily large, the function value $f(x)$ tends to (in fact, get's arbitrarily close to) the number L. Similarly,

  $$\lim_{x\to-\infty} f(x) = L$$

  means that as x gets more and more negative, the function value $f(x)$ tends to the number L. The notation

  $$\lim_{x\to\infty} f(x) = \infty$$

  is used to indicate that the function becomes arbitrarily large as x becomes large (and similarly for the limits involving $-\infty$).

  We're interested in the behavior of the function at its extremes, far from 0. This is very common in many areas of business or science: in astronomy, for example, when we're looking at the effects of Earth's gravity on a spaceship near the sun (it's essentially negligible - i.e., Earth's gravity (at great distance) is about 0: $\lim_{x\to\infty} g(x) = 0$).

- **horizontal asymptote**: if either of the two limits above hold, then the line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$.

- Precise definitions of these limits are also provided:
  - Let f be a function defined on (a, ∞). Then

    $$\lim_{x\to\infty} f(x) = L$$

    means that for every $\varepsilon > 0$ there is a corresponding number N such that

    $$|f(x) - L| < \varepsilon \text{ whenever } x > N.$$

  - Let f be a function defined on $(-\infty, a)$. Then

    $$\lim_{x\to-\infty} f(x) = L$$

    means that for every $\varepsilon > 0$ there is a corresponding number N such that

    $$|f(x) - L| < \varepsilon \text{ whenever } x < N.$$

– Let f be a function defined on (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every $M > 0$ there is a corresponding number N such that

$$f(x) > M \text{ whenever } x > N.$$

And similarly for other limits, such as $\lim_{x \to \infty} f(x) = -\infty$, etc.

b. Theorems

- If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If, furthermore, x^r is defined for negative numbers, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

c. Properties/Tricks/Hints/Etc.

- Most limit laws hold if $x \to a$ is replaced by $x \to \infty$ or $x \to -\infty$.
- A graph may tend toward a horizontal asymptote in several different ways (e.g. directly from above, directly from below, or by oscillating back and forth).
- To evaluate limits of rational functions as $x \to \pm\infty$, simply divide numerator and denominator by the highest power of x, and evaluate the limit.

d. Summary

Section 3.4 provides us with intuitive and formal definitions of infinite limits (that is, as the independent variable x tends toward negative or positive ∞). The notion of a horizontal asymptote is introduced, which is a horizontal line towards which the function tends. This allows us to effectively replace a function, which may be rather complicated, by a number when we get to extreme values of x.

A related example of this type of approximation is our day-to-day use of the force of gravity: it's usually given as 9.81 meters per second squared, and we don't generally ask our distance from the center of the Earth because the function changes so little for these modest changes in distance (hundreds of feet, say). Gravity falls off as you get farther from the surface of the Earth, toward zero, but this change is so gradual that we treat this force as a constant. If we got far enough away from the Earth, we might treat its force of gravity as another constant - i.e. 0. That is **the limit** of Earth's gravity as we get "infinitely far" away (good luck with that).