Section Summary: 2.5

a. Theorems

- **The Chain Rule** If f and g are both differentiable, and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, the F is differentiable and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

This is the secret to differentiating composite functions. You definitely need to memorize this formula.

- **The Power Rule combined with the Chain Rule** If n is any real number and $g(x)$ is differentiable, then

$$(g(x)^n)' = ng(x)^{n-1}g'(x)$$

This is simply an example of the chain rule, where the outer function is a power function. It's such an important and common example, however, that you should consider memorizing it separately.

b. Summary

The chain rule is the secret to differentiating compositions of functions, and this is a terribly important rule which you must memorize and understand.

The hardest thing about the chain rule is probably identifying the composition of functions. Given an expression, e.g. $\sin(2x-1)$, you need to realize that $f(x) = \sin(x)$, and $g(x) = 2x - 1$ (then apply the rule correctly, of course:

$$(\sin(2x-1))' = \cos(g(x))g'(x) = \cos(2x-1)2 = 2\cos(2x-1)$$

Sometimes we talk about "outer function" and "inner function". The inner function is the first function x meets on its transformation. The inner function returns a value u, which serves as the input to the outer function which returns a value y. The composite function of inner and outer thus takes a value x and returns a value y.

c. Another perspective:

Suppose that we have dependent variable y defined as a function of independent variable x

$$y = f(x)$$

but that x is itself a variable of t: $x = g(t)$. Then we can actually think of y as a function of t, as a composition

$$y = f(g(t))$$

If we want to calculate the derivative $\frac{dy}{dt}$, we can proceed in different ways:

i. We could find the formula g in $x = g(t)$, then substitute for the variable x in the formula $f(x)$. Example:

$$y = \sin(x); \text{ but } \; x = 3t; \text{ hence } \; y = \sin(3t)$$

You see that x has been eliminated from the picture. But now we have a composition - what next?

ii. Alternatively, we could reason as follows:

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta x} \frac{\Delta x}{\Delta t}$$

(one of my favorite tricks - multiply by an appropriate form of 1), so that

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta x} \frac{dx}{dt}$$

Now, if

$$\lim_{\Delta t \to 0} \Delta x = 0,$$

then

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta x} \frac{dx}{dt} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \frac{dx}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

In the example from above,

$$y = \sin(x) \text{ and } \; x = 3t; \text{ hence } \; \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \cos(x)3 = 3\cos(x)$$

and if we want to express this in terms of t, we can substitute back in for x as a function of t at the end:

$$\frac{dy}{dt} = 3\cos(3t)$$