Integration (Numerical!)

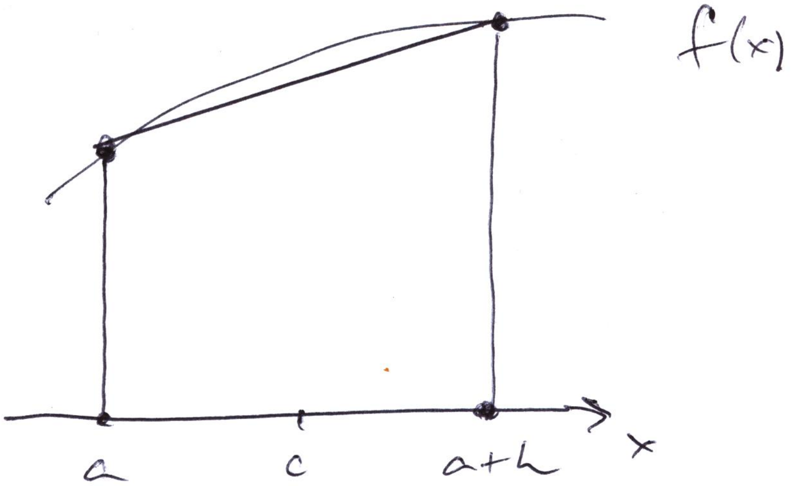
With errors... :(

But we knew that! :)

Big picture: $\int_a^b f(x)dx$

But $[a,b]$ is likely so big, & f is so active on it, that we'll have to break it up (<u>partition</u> it) into sub-intervals, which we continue to do (adaptively – based on the function itself) until the sub-intervals are small enough that the simple methods (trap, mid, simp) work well.

So what kinds of errors do we make using trapezoidal?



$$\int_a^{a+h} f(x)\,dx \simeq \text{width} * \text{average height}$$

$$\simeq \frac{h}{2}\left(f(a+h) + f(a)\right)$$

$$E_T = -\int_a^{a+h} f(x)\,dx + \frac{h}{2}\left(f(a+h) + f(a)\right)$$

$$= \left[\frac{h}{2}\left(f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + f(a) \right) - \int_a^{a+h} f(a+(x-a))\,dx \right]$$

$$= \left[hf(a) + \frac{h}{2}\left(hf'(a) + \frac{h^2}{2!} f''(a) + \cdots \right) - \int_a^{a+h} \left(f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2} f''(a) + \cdots \right) dx \right.$$

$$= \Big[\cancel{hf(a)} + \frac{h}{2}\left(hf'(a) + \frac{h^2}{2!}f''(a) + \dots\right)$$

$$- \left(\cancel{hf(a)} + \frac{(x-a)^2}{2} f'(a) \Big|_a^{a+h} + \frac{(x-a)^3}{3!} f''(a) \Big|_a^{a+h} + \dots \right) \Big]$$

$$= \frac{h}{2}\left(hf'(a) + \frac{h^2}{2!}f''(a) + \dots\right)$$

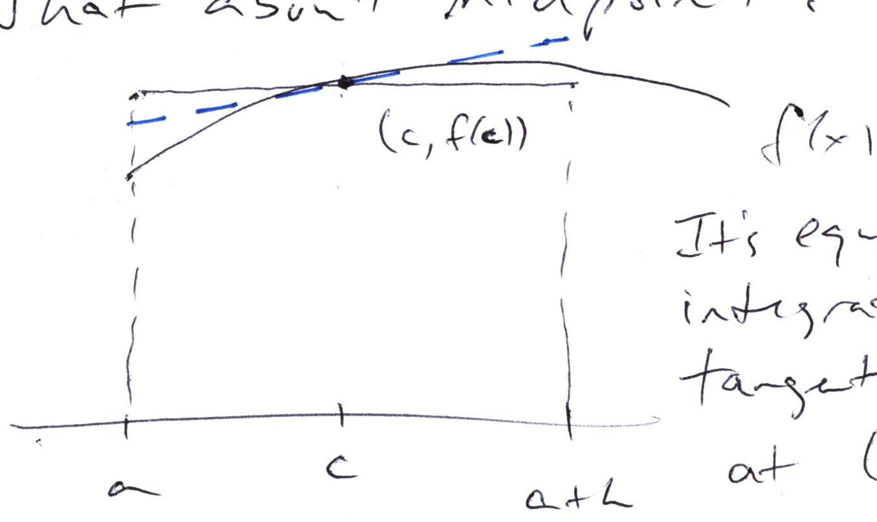
$$- \frac{h^2}{2} f'(a) - \frac{h^3}{3!} f''(a) - \dots$$

$$= \left(\frac{h^3}{4} - \frac{h^3}{6}\right) f''(a) + O(h^4)$$

$$= \boxed{\frac{h^3}{12} f''(a) + O(h^4)}$$

So this is one picture of the error we make using trapezoidal.

What about midpoint?



It's equivalent to integrating the tangent function at $(c, f(c))$

$$M = \text{width} * \text{height}$$
$$= h \cdot f(c) \simeq I = \int_a^{a+h} f(x)\,dx$$

$$E_M = \underbrace{h f(c)} - \int_a^{a+h} f(x)\,dx$$

Taylor Series about $x=a$, so we can compare with trapezoidal

$$= h f(a + \underbrace{(c-a)}_{h/2}) - \int_a^{a+h} f(x)\,dx$$

$$= h\left(\cancel{f(a)} + \cancel{\tfrac{h}{2} f'(a)} + \frac{(\frac{h}{2})^2}{2} f''(a) + \cdots\right)$$

(from before) $$- \cancel{h f(a)} - \cancel{\tfrac{h^2}{2} f'(a)} - \frac{h^3}{3!} f''(a) - \cdots$$

$$= \frac{h^3}{8} f''(a) - \frac{h^3}{6} f''(a) + O(h^4)$$

$$= \boxed{\frac{-h^3}{24} f''(a) + O(h^4)}$$

The picture of midpoint's error to compare to trapezoidal

Comparison of these errors suggests Simpson's rule - an average of two good methods, but a weighted average:

$$S = \frac{2M + T}{3} \simeq I$$

The errors of M & T balance (and cancel) to give a higher order method. Actually it $O(h^5)$ & proportional to $f^{(v)}(c)$ -

It gets cubics exactly <u>right</u>!

$$S = \frac{2hf(c) + \frac{h}{2}(f(a) + f(a+h))}{3}$$

$$= h\left[\frac{1}{6}f(a) + \frac{2}{3}f(c) + \frac{1}{6}f(a+h)\right]$$

$$= h\left[\frac{1}{6}f(a) + \underbrace{\frac{4}{6}f(c)}_{\text{four times the weight}} + \frac{1}{6}f(a+h)\right]$$

Geometrically, S can be thought of as the integral of the quadratic interpolator of

$(a, f(a))$
$(c, f(c))$
$(a+h, f(a+h))$