We want to prove Theorem 3.5, p99:
The order of convergence of the secant method is $\varphi \simeq 1.618$.

Recall:

(1) $$x_{k+1} = x_k - \frac{f(x_k)}{\frac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}}}$$

Assume convergence to root r:

$$\lim_{k\to\infty} x_k = r$$

Define $e_k = x_k - r$ (the error)

(1) $$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k)-f(x_{k-1})}$$

Subtract r from both sides:

$$\underbrace{x_{k+1} - r} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k)-f(x_{k-1})} - r$$

$$e_{k+1} = \frac{x_k(f(x_k)-f(x_{k-1})) - f(x_k)(x_k - x_{k-1}) - r(f(x_k)-f(x_{k-1}))}{f(x_k)-f(x_{k-1})}$$

2

$$e_{k+1} = \frac{-x_k f(x_{k-1}) + x_{k-1} f(x_k) - r f(x_k) + r f(x_{k-1})}{f(x_k) - f(x_{k-1})}$$

$$= \frac{(x_{k-1} - r) f(x_k) - (x_k - r) f(x_{k-1})}{f(x_k) - f(x_{k-1})}$$

$$= \frac{e_{k-1} f(x_k) - e_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}$$

$$= \frac{e_{k-1}\left(f(r) + f'(r) e_k + \frac{f''(\xi_k)}{2} e_k^2\right) - e_k\left(f(r) + f'(r) e_{k-1} + \frac{f''(\xi_{k-1})}{2} e_{k-1}^2\right)}{\left(f(r) + f'(r) e_k + \frac{f''(\xi_k)}{2} e_k^2\right) - \left(f(r) + f'(r) e_{k-1} + \frac{f''(\xi_{k-1})}{2} e_{k-1}^2\right)}$$

$$e_{k+1} = \frac{e_{k-1} e_k \left(\frac{f''(\xi_k)}{2} e_k - \frac{f''(\xi_{k-1})}{2} e_{k-1}\right)}{f'(r)(e_k - e_{k-1}) + \frac{f''(\xi_k)}{2} e_k^2 - \frac{f''(\xi_{k-1})}{2} e_{k-1}^2}$$

$$\frac{e_{k+1}}{e_k e_{k-1}} = \frac{\frac{f''(\xi_k)}{2} e_k - \frac{f''(\xi_{k-1})}{2} e_{k-1}}{f'(r)(e_k - e_{k-1}) + \frac{f''(\xi_k)}{2} e_k^2 - \frac{f''(\xi_{k-1})}{2} e_{k-1}^2}$$

What if $e_k = e_{k-1}$?

Then $x_k = x_{k-1}$

& $\xi_k = \xi_{k-1}$

③

$$\frac{e_{k+1}}{e_k e_{k-1}} = \frac{(e_k - e_{k-1})\left[\dfrac{\frac{f''(\xi_k)}{2} e_k - \frac{f''(\xi_{k-1})}{2} e_{k-1}}{e_k - e_{k-1}}\right]}{(e_k - e_{k-1})\left[f'(r) + \dfrac{\frac{f''(\xi_k)}{2} e_k^2 - \frac{f''(\xi_{k-1})}{2} e_{k-1}^2}{e_k - e_{k-1}}\right]}$$

$$= \frac{\dfrac{\frac{f''(\xi_k)}{2} e_k - \frac{f''(\xi_{k-1})}{2} e_{k-1}}{e_k - e_{k-1}}}{f'(r) + \dfrac{\frac{f''(\xi_k)}{2} e_k^2 - \frac{f''(\xi_{k-1})}{2} e_{k-1}^2}{e_k - e_{k-1}}}$$

Now pass to the limit:

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k e_{k-1}} = \frac{\lim_{k\to\infty} \dfrac{f''(\xi_k) e_k - f''(\xi_{k-1}) e_{k-1}}{2(e_k - e_{k-1})}}{\lim_{k\to\infty} f'(r) + \underbrace{\lim_{k\to\infty} \dfrac{\frac{f''(\xi_k)}{2} e_k^2 - \frac{f''(\xi_{k-1})}{2} e_{k-1}^2}{e_k - e_{k-1}}}_{\to 0}}$$

$$= \frac{f''(r)}{2 f'(r)}$$

Limit appears on p 100

An aside, dedicated to the ever-inquisitive Michael W (& rightly so! :).

To show that

$$① \quad \lim_{k\to\infty} \frac{f''(\xi_k)e_k - f''(\zeta_{k-1})e_{k-1}}{e_k - e_{k-1}} = f''(r)$$

my favorite trick (add the appropriate form of zero):

$$\lim_{k\to\infty} \frac{f''(\xi_k)e_k - f''(\xi_k)e_{k-1} + f''(\xi_k)e_{k-1} - f''(\zeta_{k-1})e_{k-1}}{e_k - e_{k-1}}$$

$$= \lim_{k\to\infty} \frac{f''(\xi_k)(e_k - e_{k-1}) + e_{k-1}\left(f''(\xi_k) - f''(\zeta_{k-1})\right)}{e_k - e_{k-1}}$$

$$= \lim_{k\to\infty} \left[f''(\xi_k) + \frac{e_{k-1}\left(f''(\xi_k) - f''(\zeta_{k-1})\right)}{e_k - e_{k-1}} \right]$$

$$= \lim_{k\to\infty} f''(\xi_k) + \lim_{k\to\infty} \frac{e_{k-1}\left(f''(\xi_k) - f''(\zeta_{k-1})\right)}{e_k - e_{k-1}}$$

$$= f''(r) + \lim_{k\to\infty} \frac{f''(\xi_k) - f''(\zeta_{k-1})}{\frac{e_k}{e_{k-1}} - 1}$$

Now I want to show that this 2nd term $= 0$

The numerator clearly goes to 0, since

$$\lim_{k\to\infty}\left[f''(\xi_k) - f''(\xi_{k-1})\right] = f''(r) - f''(r) = 0$$

I'll show that the denominator goes to -1, so the quotient goes to 0.

Claim: $\displaystyle \lim_{k\to\infty} \frac{e_k}{e_{k-1}} = \lim_{k\to\infty} \frac{e_{k+1}}{e_k} = 0.$

$$x_{k+1} = x_k - \frac{f(x_k)}{f(x_k)-f(x_{k-1})}\cdot(x_k - x_{k-1})$$

$$x_{k+1} - r = x_k - r - \frac{f(x_k)}{f(x_k)-f(x_{k-1})}(x_k - x_{k-1})$$

$$e_{k+1} = e_k - \frac{f(x_k)}{f(x_k)-f(x_{k-1})}(x_k - x_{k-1})$$

$$\frac{e_{k+1}}{e_k} = 1 - \frac{f(x_k)}{f(x_k)-f(x_{k-1})}\,\frac{(x_k - x_{k-1})}{e_k}$$

$$= 1 - \frac{f(x_k)}{e_k}\cdot\frac{x_k - x_{k-1}}{f(x_k)-f(x_{k-1})}$$

$$= 1 - \frac{f(x_k)-f(r)}{x_k - r}\cdot\frac{x_k - x_{k-1}}{f(x_k)-f(x_{k-1})}$$

$$\lim_{k\to\infty}\frac{e_{k+1}}{e_k} = 1 - \lim_{k\to\infty}\frac{f(x_k)-f(r)}{x_k - r}\cdot\lim_{k\to\infty}\frac{x_k - x_{k-1}}{f(x_k)-f(x_{k-1})}$$

$$= 1 - f'(r)\cdot\frac{1}{f'(r)} = 0$$

$\therefore \lim_{k\to\infty}\left(\frac{e_k}{e_{k-1}} - 1\right) = \lim_{k\to\infty} \frac{e_k}{e_{k-1}} - \lim_{k\to\infty} 1$

$= \lim_{k\to\infty} \frac{e_{k+1}}{e_k} - 1 = 0 - 1 = -1.$

So

$$\text{(1)} \quad \lim_{k\to\infty} \frac{f''(\xi_k)e_k - f''(\xi_{k-1})e_{k-1}}{e_k - e_{k-1}} = f''(r)$$

(you can see why I didn't want to "go there" in class!)

There is that denominator term that I claimed goes to 0:

$$\lim_{k\to\infty} \frac{\frac{f''(\xi_k)}{2}e_k^2 - \frac{f''(\xi_{k-1})}{2}e_{k-1}^2}{e_k - e_{k-1}} = 0$$

This is because we have quadratic terms e_k^2 + e_{k-1}^2 in the numerator, rather than the linear terms e_k + e_{k-1} in (1). This drives the term to 0.

Now back to our regular programming... ☺

We want the order of convergence:

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k|^p} = C \qquad (\text{we want } p)$$

We know that

$$\lim_{k\to\infty}\left|\frac{e_{k+1}}{e_k e_{k-1}}\right| = \left|\frac{f''(r)}{2f'(r)}\right| \quad (\equiv M)$$

$$\lim_{k\to\infty}\left|\frac{e_{k+1}}{e_k^{p}}\right| = |C|$$

Look at the ratio:

$$\lim_{k\to\infty}\left|\frac{\dfrac{e_{k+1}}{e_k e_{k-1}}}{\dfrac{e_{k+1}}{e_k^{p}}}\right| = \left|\frac{M}{C}\right|$$

$$= \lim_{k\to\infty}\left|\frac{e_k^{p-1}}{e_{k-1}}\right| = \left|\frac{M}{C}\right|$$

So $$\lim_{k\to\infty}\left|\frac{e_{k+1}^{p-1}}{e_k}\right| = \left|\frac{M}{C}\right|$$ as well.

and $$\lim_{k\to\infty}\left|\frac{e_{k+1}}{e_k^{p}}\right| = |C|$$

So $$\lim_{k\to\infty}\left|\frac{e_{k+1}^{1/p}}{e_k}\right|^{p} = |C|$$

and
$$\lim_{k\to\infty}\left|\frac{e_{k+1}}{e_k}\right|^{1/p} = |C|^{1/p}$$

also
$$\lim_{k\to\infty}\left|\frac{e_{k+1}}{e_k}\right|^{p-1} = \left|\frac{M}{C}\right|$$

$$\therefore \quad 1/p = p-1$$

$$1 = p^2 - p$$

$$0 = p^2 - p - 1$$

$$p = \frac{1 \pm \sqrt{5}}{2} \quad (\text{Take } +)$$

$$p = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$$

Furthermore
$$|C|^{1/p} = \left|\frac{M}{C}\right|$$