$$A(n) = A(n-1) + A(n-2) + 1$$

Change of variables:

$$B(n) = A(n) - c$$

$$A(n) = B(n) + c$$

$$B(n) + \cancel{c} = B(n-1) + \cancel{c} + B(n-2) + c + 1$$

$$B(n) = B(n-1) + B(n-2) + c + 1$$

Choose c so that

$$c + 1 = 0$$

$$c = -1$$

$$A(n) = B(n) - 1$$

Take a problem we don't know how to do - constant coefficient, <u>non-homogeneous</u> recurrence relation, second-order - & cast it back as a <u>homogeneous</u> problem - solve the homogeneous one, & bring the solution back.

Suppose we had this sort of recurrence relation:

$$A(n) = A(n-1) \cdot A(n-2)$$

non-linear

& assume the A are positive, & we know $A(1)$ & $A(2)$.

Take logs of both sides:

$$\ln A(n) = \ln\left(A(n-1)\, A(n-2)\right)$$

$$= \ln(A(n-1)) + \ln(A(n-2))$$

Define $B(n) = \ln(A(n))$

Now it's constant coefficient, linear, 2nd order, etc...

$$B(n) = B(n-1) + B(n-2)$$

$$A(n) = A(n-1)A(n-2)$$

$A(0) = 1$

$A(1) = 2$

$A(3) = 2$

$A(4) = 4$

$A(5) = 8$

$A(6) = 16$

$$A(n) = 2^{F(n)}$$

$$\log(A(n)) = \log(A(n-1)) + \log(A(n-2))$$

$$B(n) = B(n-1) + B(n-2)$$

$B(0) = 0$ $\qquad \log(A(0)) = 0$

$B(1) = 1$ $\qquad \log(A(1)) = 1$

$B(2) = 1$ $\qquad \ln(A(2)) = 1$

$B(3) = 2$

$B(4) = 3$

$B(5) = 5$

$\vdots$

$$B(n) = F(n)$$

$$A(n) = 2^{F(n)}$$

A

#20 p179 $\qquad n = 2^m$

$$C(n) = 2C\left(\tfrac{n}{2}\right) + \left(\tfrac{n}{2}\right) + \left(\tfrac{n}{2}\right) - 1$$

$$\begin{cases} C(n) = 2C\left(\tfrac{n}{2}\right) + n - 1 \\ C(1) = 0 \end{cases}$$

$$C(n) = \cancel{c^{\log n} C(1)} + \sum_{i=1}^{\log n} c^{\log n - i} g(2^i)$$

$$c = 2 \qquad g(n) = n - 1$$

$$C(n) = \sum_{i=1}^{\log n} 2^{\log n - i}\left(2^i - 1\right)$$

$$= \sum_{i=1}^{\log n} 2^{\log n - i} \cdot 2^i - \sum_{i=1}^{\log n} 2^{\log n - i}$$

$$= \sum_{i=1}^{\log n} 2^{\log n} - \left(2^0 + 2^1 + \dots + 2^{\log n - 1}\right)$$

$$= \sum_{i=1}^{\log n} n - \frac{1 - 2^{\log n}}{1 - 2}$$

$$\boxed{C(n) = n \log n + 1 - n}$$

This is what the sort costs.

So binary search is more costly than sequential search if we have to add in the cost of the sort! $n \log n$ versus

n for sequential search.

Euclidean algorithm for the greatest common divisor, gcd.

Example: gcd(27, 15)

27 and 15

$27 = 1 \cdot 15 + 12$

$15 = 1 \cdot 12 + \boxed{3}$

$12 = 4 \cdot 3 + 0$

In general: we want gcd(a, b) where $a > b$ (a, b are natural numbers).

$a = q_1 b + r_1$ $\quad r_1 < b$

$b = q_2 r_1 + r_2$ $\quad r_2 < r_1$

$r_1 = q_3 r_2 + r_3$ $\quad r_3 < r_2$

$r_1 = \left\lfloor \frac{r_1}{r_2} \right\rfloor r_2 + r_3$ $\quad \lfloor \; \rfloor$ floor function

rewrite as

$r_3 = r_1 - \left\lfloor \frac{r_1}{r_2} \right\rfloor r_2$

$$\boxed{r_n = r_{n-2} - \left\lfloor \frac{r_{n-2}}{r_{n-1}} \right\rfloor r_{n-1}}$$

non-linear recurrence relation

(Second order), homogeneous.

We're interested in worst-case behavior - i.e., it goes slowly. So r_n decreases, but slowly. We can make that happen by making (demanding!)

$$\left\lfloor \frac{r_{n-2}}{r_{n-1}} \right\rfloor = 1 .$$

So analyze a sister recurrence relation,

$$r_n = r_{n-2} - r_{n-1}$$

$$r_{n-2} = r_{n-1} + r_n$$

Looks like Fibonacci's.

The smallest a that leads to <u>one division</u> is $a = 2 : b = 1$.

$$2 = 2 \cdot 1 + 0$$

$$r_m = 0$$

$$r_{m-1} = 1 \quad = F(1)$$

$$r_{m-2} = 1 \quad = F(2)$$

$$r_{m-3} = 2 \quad = F(3)$$

$$r_{m-4} = 3 \quad = F(4)$$

$$r_{m-5} = 5 \quad = F(5)$$

$$\vdots$$

$$r_0 \quad = F(m)$$

$$b \quad = F(m+1)$$

$$a \quad = F(m+2)$$

$\gcd(F(m+2), F(m+1))$ requires m divisions

#26 Suppose that m divisions are required to find $\gcd(a,b)$, w/ $m \geq 4$ and $a = n$. Prove that

$$\left(\tfrac{3}{2}\right)^{m+1} < F(m+2) \leq n$$

(p141) #23: $F(n) > \left(\tfrac{3}{2}\right)^n$ for $n \geq 6$

$$\rightarrow \quad F(m+2) > \left(\tfrac{3}{2}\right)^{m+2} \quad \text{for } m \geq 4$$

$$\therefore \left(\tfrac{3}{2}\right)^{m+1} < \left(\tfrac{3}{2}\right)^{m+2} < F(m+2) \quad \text{for } m \geq 4.$$

(p180) #25 $n = a \geq F(m+2)$

$$\therefore \boxed{\left(\tfrac{3}{2}\right)^{m+1} < F(m+2) \leq n}$$

#27 Suppose m divisions are required to find $\gcd(a,b)$ w/ $m \geq 4$ and $a = n$. Prove that

$$m < \log_{1.5} n - 1$$

$$\left(\tfrac{3}{2}\right)^{m+1} \leq n$$

$$\log_{3/2}\left[\left(\tfrac{3}{2}\right)^{m+1}\right] \leq \log_{3/2} n$$

$$m+1 \leq \log_{1.5} n \quad \rightarrow \quad \boxed{m < \log_{1.5} n - 1}$$

#28 a. gcd(89, 55)

gcd(55, 34)

" (34, 21)

(21, 13)

(13, 8) — 9 divisions

(8, 5)

(5, 3) — $n < \log_{1.5} 89 - 1$

(3, 2)

gcd(2, 1)

b. $E(n) \leq 2 \log_2 89 < 13$

12.95

c. $E(n) < \log_{1.5} 89 - 1 = 10.07$

$\boxed{9}$