Motel Infinity
(when it's full, & the $\mathbb{N}$ school bus pulls up):

Room : $n \to 2n$

Bus : $n \to 2n+1$

Two copies of $\mathbb{N}$ fit
into $\mathbb{N}$

Motel empty

Bus 1 : $n \to 2^n$

Bus 2 : $n \to 3^n$

Bus 3 : $n \to 5^n$

$\vdots \quad \vdots$

Prime factorization Theorem guarantees that there's a unique factorization of each natural # into a product of primes.

To determine that two (in)finite sets are the same size you construct a 1-1 (one to one) mapping between the sets.

So the natural numbers & the even natural numbers are the same size:

$$\begin{array}{ccc} \mathbb{N} & & \text{Evens} \\ n & \longleftrightarrow & 2n \end{array}$$

(they have the same <u>cardinality</u>)

Suppose that someone claims that $\mathbb{N}$ is the same size as $\mathcal{P}(\mathbb{N})$. Then there is a 1-1 correspondence between them. Let's call it $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

$\mathbb{N}$		$\mathcal{P}(\mathbb{N})$
$f(1)$	$\longleftrightarrow$	evens
$f(2)$	$\longleftrightarrow$	odds
$f(3)$	$\longleftrightarrow$	primes
$f(4)$	$\longleftrightarrow$	$\{1,4\}$

etc. $\downarrow$

Every natural # has its partner in $\mathcal{P}(\mathbb{N})$ at the dance - and vice versa.

So we construct a set, A, which is not at the dance, as follows:

$$A = \{x \in S \mid x \notin f(x)\}.$$

For the example above, we would start

with $x = 1$: $1 \notin$ evens, so

$1 \in A$: $A = \{1, \ldots\}$

$x = 2$: $2 \notin$ odds, so

$2 \in A$: $A = \{1, 2, \ldots\}$

$x = 3$: $3 \in$ primes, so

$3 \notin A$: $A = \{1, 2, \ldots\}$

$x = 4$: $4 \in \{1, 4\}$, so

$4 \notin A$: $A = \{1, 2, \ldots\}$

The whole point is to make sure that A is different from ~~at least one~~ every member of $\mathcal{P}(\mathbb{N})$, in at least one place. Since it's different from every dance partner, it couldn't have been given a partner for the dance.

And since $A \in \mathcal{P}(\mathbb{N})$, that means that the 1-1 correspondence didn't really exist. Hence $\mathcal{P}(\mathbb{N})$ is bigger, since $\mathbb{N}$ couldn't match up to it.