Directions: Problems are equally weighted. Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

Problem 1: (10 pts) Propositional logic: Let A, B, C, and D be the following statements:

- A: The villain is French.
- B: The hero is American.
- C: The heroine is British.
- D: The movie is good.

Translate the following compound statements into symbolic notation:

a. The hero is American and the movie is good.

$\boxed{B \wedge D}$

b. The hero is not American, but the villain is French.

$\boxed{B' \wedge A}$

c. If the movie is good, then either the hero is American or the heroine is British.

$\boxed{D \rightarrow B \vee C}$

d. The villain is French is a sufficient condition for the hero to be American.

$\boxed{A \rightarrow B}$

e. A British heroine is a necessary condition for the movie to be good.

$\boxed{D \rightarrow C}$

Problem 2: (10 pts) "If interest rates drop, the housing market will improve. Either the federal discount rate will drop or the housing market will not improve. Interest rates will drop. Therefore the federal discount rate will drop." Use I (interest rates will drop), H (the housing market will improve), and F (the federal discount rate will drop).

a. (4 pts) Write the theorem.

$(I \to H) \wedge (F \vee H') \wedge I \to F$

b. (3 pts) Prove it using propositional logic.

1.	$I \to H$	hyp
2	$F \vee H'$	hyp
3	I	hyp
4	H	1,3, mp
5	F	2,4, ds

c. (3 pts) Check the result using a truth table, which I've started here:

I	H	F	$(I \to H)$	$(F \vee H')$	$(I \to H) \wedge (F \vee H') \wedge I \to F$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Problem 3: (10 pts) Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is "the whole world".)

- $G(x)$: x is a game.
- $M(x)$: x is a movie.
- $F(x, y)$: x is more fun than y.

a. Any movie is more fun than any game.

$(\forall x)[M(x) \to (\forall y)[G(y) \to F(x,y)]]$ ✓

b. No game is more fun than every movie.

$(\forall x)[G(x) \to (\exists y)[M(y) \wedge F(y,x)]]$ ✓

c. Only games are more fun than movies.

$(\forall x)(\forall y)[M(x) \wedge F(y,x) \to G(y)]$ ✓

d. All games are more fun than some movie.

$(\exists x)[M(x) \wedge (\forall y)[G(y) \to F(y,x)]]$ ✓

That's a different meaning than intended, but I get it.

10

Problem 3: (10 pts) Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is "the whole world".)

- $G(x)$: x is a game.
- $M(x)$: x is a movie.
- $F(x, y)$: x is more fun than y.

a. Any movie is more fun than any game.

$$(\forall x)[M(x) \rightarrow (\forall y)[G(y) \rightarrow F(x,y)]]$$

b. No game is more fun than every movie.

$$(\forall x)[G(x) \rightarrow (\exists y)[M(y) \wedge F(y,x)]]$$

c. Only games are more fun than movies. $\not\Leftrightarrow$ "Movies are more fun than anything that isn't a game"

$$(\forall x)[(G(x))' \rightarrow (\forall y)[M(y) \rightarrow F(y,x)]]$$

No - what if x & y are both movies?

Good effort

d. All games are more fun than some movie.

$$(\forall x)[G(x) \rightarrow (\exists y)[M(y) \wedge F(x,y)]]$$

Problem 4: (10 pts) I read the New York Times yesterday, and learned that Congress wants to **define** milk as "lacteal secretion of hooved mammals." This got me to thinking: Congress has decided that if a child drinks its mother's milk, then its mother is a hooved mammal – Go Congress!

Assume the following, and use it to prove that there is a person who is a hooved animal:

- s is my son;
- m is my son's mother;
- My son was breast-fed by his mother;
- If one person gives its milk to another, then that milk-giver must be a hooved animal;

Use the letters P (person), H (hooved), G(x,y) (x gives y its milk).

a. (5 pts) Properly state the theorem using predicate logic,

$(\forall x)(\forall y)[P(x) \wedge P(y) \wedge G(x,y) \to H(x)] \wedge P(s) \wedge P(m) \wedge G(m,s) \to (\exists z)[P(z) \wedge H(z)]$

good

b. (5 pts) Prove the theorem.

1. $P(s)$
2. $P(m)$ hyp
3. $G(m,s)$ hyp
4. $(\forall x)(\forall y)[P(x) \wedge P(y) \wedge G(x,y) \to H(x)]$ hyp
5. $(\forall y)[P(m) \wedge P(y) \wedge G(m,y) \to H(m)]$ 4, ui
6. $P(m) \wedge P(s) \wedge G(m,s) \to H(m)$ 5, ui
7. $P(m) \wedge P(s)$ 2, 1 conj.
8. $P(m) \wedge P(s) \wedge G(m,s)$ 7, 3 conj.
9. $H(m)$ 6, 8 mp
10. $P(m) \wedge H(m)$ 2, 9 conj
11. $(\exists z)[P(z) \wedge H(z)]$ 10, eg

excellent

roblem 5: (10 pts) Prove the theorem

$$A \wedge (B \to C) \to (B \to (A \wedge C))$$

in three different ways:

a. Directly:

1. A hyp
2. $B \to C$ hyp
3. B hyp (deduction)
4. C 2, 3, mp
5. $A \wedge C$ 1, 4, conj

b. By contraposition:

$(B \to (A \wedge C))' \to (A \wedge (B \to C))'$

$(B \to (A \wedge C))' \to (A' \vee (B \to C)')$ De Morgan on the right side

c. By contradiction:

$A \wedge (B \to C) \wedge (B \to (A \wedge C))' \to \text{false}$

1. A hyp
2. $B \to C$ hyp
3. $(B \to (A \wedge C))'$ hyp
4. B temp hyp
5. C 4, mp
6. $A \wedge C$ 1, 5, conj
7. $B \to (A \wedge C)$ disch

good

3, 7

4. $B' \vee (A \wedge C)'$ 3, imp
5.
6.

contradiction

Problem 5: (10 pts) Prove the theorem

$$A \wedge (B\to C)\to(B\to(A\wedge C))$$

$A \wedge (B\to C) \wedge B \to (A\wedge C)$

in three different ways:

a. Directly:

1. A hyp
2. $B\to C$ hyp
3. B hyp
4. C 3,2 mp
5. $A\wedge C$ 1,4, conj

b. By contraposition:

$(B' \vee (A\wedge C))' \to (A\wedge(B'\vee C))'$

$B\wedge(A' \wedge C') \to (A' \vee (B\wedge C'))$

$B\wedge A' \wedge C' \to (\underbrace{A \to B\wedge C'}_{\wedge A \to B\wedge C'})$

1. B hyp
2. A' hyp
3. C' hyp
4. A exp
5. $B\wedge C'$ 1,3, conj

inconsistency

-.5

good work

c. By contradiction:

$A\wedge(B\to C)\wedge B\wedge (A\wedge C)' \to 0$

$A\wedge(B\to C)\wedge B\wedge(A'\vee C')$

1. A hyp
2. $B\to C$ hyp
3. B hyp
4. $A'\vee C'$ hyp
5. C 2,3, mp
6. C' 4,1 ds

inconsistency

Problem 5: (10 pts) Prove the theorem

$$A \wedge (B \to C) \to (B \to (A \wedge C))$$

in three different ways:

a. Directly:

1	A	hyp
2	$B \to C$	hyp
3.	B	hyp
4	C	mp, 3,2
5	$A \wedge C$	conj, 4, 1

b. By contraposition: $Q' \to P'$

$(B \to (A \wedge C))' \to (A \wedge (B \to C))'$

~~$A \wedge (B \to C) \to (A \wedge C)' \to B'$~~

~~$B' \to (A \wedge C)' \to (A' \vee (B \to C)')$~~ ~~$A \wedge (B \to C) \to A' \vee C' \to B'$~~

A	B	C	$A \wedge C$	$B \to (A \wedge C)$	$(B \to (A \wedge C))'$	$(B \to C)$	$A \wedge (B \to C)$	$'$	$(B \to (A \wedge C))' \to (A \wedge (B \to C))'$
T	T	T	T	T	F	T	T	F	T
T	T	F	F	F	T	F	F	T	T
T	F	T	T	T	F	T	T	F	T
T	F	F	F	T	F	T	T	F	T
F	F	F	F	T	F	T	F	T	T
F	F	T	F	T	F	T	F	T	T
F	T	F	F	F	T	F	F	T	T
F	T	T	F	F	T	T	F	T	T

c. By contradiction: $B \to (A \wedge C)$ is false so $(B \to (A \wedge C))'$ is true

A	B	C	$B \to C$	$A \wedge (B \to C)$	$A \wedge C$	$B \to (A \wedge C)$	$'$	$(A \wedge (B \to C) \to (B \to (A \wedge C)))'$
T	T	T	T	T	T	T	F	F
T	T	F	F	F	F	F	T	T
T	F	T	T	T	T	T	F	F
T	F	F	T	T	F	T	F	F
F	F	F	T	F	F	T	F	T
F	F	T	T	F	F	T	F	T
F	T	F	F	F	F	F	T	T
F	T	T	T	F	F	T	F	T

as expected, we found contradictions

No – we should get F down the line!

–.5

But good effort!

9.5

Problem 7: (10 pts) Given my fondness for Fibonacci numbers, you shouldn't be surprised to see a problem involving them.

a. Prove the property $F(n+3) = 2F(n+1) + F(n)$ for $n \geq 2$. You may do so directly, from the recurrence relation, or using induction.

$$F(n+3) = 2F(n+1) + F(n)$$

$$F(n) = f(n-1) + f(n-2)$$
$$F(n+1) = f(n) + f(n-1)$$
$$f(n+2) = f(n+1) + f(n)$$
$$f(n+3) = f(n+2) + f(n+1)$$
$$= f(n+1) + f(n) + f(n+1)$$
$$= 2f(n+1) + f(n)$$

✓

b. Here is a variation on the Fibonacci recurrence: write the correct recurrence relation for rabbit pairs, if it takes **two** months for rabbit pairs to mature before they begin producing **two** new pairs a month.

m(x) = months

extra month

m(1) = 1 pair
m(2) = 1 pair
m(3) = 1 mature pair + 2 new pairs = 3
m(4) = 1 mature pair + 2 adolence pairs + 2 new pairs = 5
m(5) = 3 mature pairs + 2 adolence pairs + 6 new pairs = 11
m(6) = 5 mature pairs + 6 adolence pairs + 10 new pairs = 21

$$\boxed{P(n) = P(n-1) + 2P(n-2)}$$

P(6) = 11 + 10 = 21

You've got the 2 but missed the extra month

-1

Problem 7: (10 pts) Given my fondness for Fibonacci numbers, you shouldn't be surprised to see a problem involving them.

a. Prove the property $F(n+3) = 2F(n+1) + F(n)$ for $n \geq 2$. You may do so directly, from the recurrence relation, or using induction.

~~F(2)~~ Base $n=2$

$F(2+3) = 2F(2+1) + F(2)$

$F(5) = 2F(3) + F(2)$ fib. def.

$F(5) = F(3) + F(4)$ fib. def.

$F(5) = F(5)$ ✓

need 2nd principle

Inductive

$F(n+3) = 2F(n+1) + F(n)$

$F((n+1)+3) = 2F((n+1)+1) + F(n+1)$

$F(n+4) = 2F(n+2) + F(n+1)$

$F(n+4) = F(n+3) + F(n+2)$

$F(n+4) = 2F(n+1) + F(n) + F(n+2)$ hyp

$F(n+4) = F(n+2) + F(n+3)$ fib. def.

$F(n+4) = F(n+4)$ ✓

b. Here is a variation on the Fibonacci recurrence: write the correct recurrence relation for rabbit pairs, if it takes **two** months for rabbit pairs to mature before they begin producing **two** new pairs a month.

$S(1) = m$

$S(2) = m$

$S(3) = M$

$S(4) = M, m, m$

$S(5) = M, m, m, m, m$

$S(6) = M, M, M, m, m, m, m$

$S(7) = M, M, M, M, M, m, m, m, m, m, m, m, m$

$S(1) = 1$

$S(2) = 1$

$S(3) = 1$

$S(4) = 2S(1) + S(3)$

$S(5) = 2S(2) + S(4)$

$S(6) = 2S(3) + S(5)$

$S(7) = 2S(4) + S(6)$

$S(n) = 2S(n-3) + S(n-1)$

Nice

Problem 8: (10 pts) Let's imagine that we have thorough system for checking cars in a manufacturing process. I first check each car myself, then I divide the cars into two groups, and pass them along to two subordinates, who check each of theirs. Then they divide their subroups into two groups, and so on – until each customer gets the final check for each car. How many times will each car be checked?

a. (5 pts) Assume that we start with $n = 2^m$ cars. Write a recurrence relation for $C(n)$, the number of checks that n cars go through. Don't forget the base case(s).

$2+1+1=4$ $n=2$

$n=8$

$8+4+4+2+2+2+2+(1)(8)=32$

$$\boxed{\begin{gathered} C(1)=1 \\ C(n)=2\cdot C\left(\tfrac{n}{2}\right)+n \\ \text{for } n\geq 2,\ n=2^m \end{gathered}}$$

$4+2+2+$
$1+1+1+1=12$
$n=4$

for base case of $n=1$ the customer can be involved from the first check

$((\log_2 n+1)(n))$

n	$C(n)$
1	1
2	4
4	12
8	32

b. (5 pts) Find a closed form solution for $C(n)$.

$g(n)=n$

$$C(n)=2^{\log n}S(1)+\sum_{i=1}^{\log n}2^{(\log n)-i}\cdot g(2^i)$$

$$=2^{\log n}S(1)+\sum_{i=1}^{\log n}2^{\log n-\cancel{i}}\cdot\cancel{2^i}$$

$$=n\cdot 1+\sum_{i=1}^{\log n}n \quad = \quad n+n\log n$$

good work!

$$=n+\frac{\log n(\log n+1)}{2}$$

$$C(n)=[(\log_2 n)+1](n)$$

I determined the closed form first and used that to work backward to the recurrance relation. Didn't have time to prove...

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