Section Summary: 4.2 - The definite integral

a. Definitions

- **definite integral of f from a to b**: The definite integral of f from a to b is defined by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where the x_i^* are sample points, one from each of the n rectangles of width Δx.

This looks like the area definition from section 4.1, but in that case the area has to be positive. Not so for the definite integral.

Integral sign: $\int$ (due to Leibniz) - an elongated S, representing a sum. Integration is just summation of an infinite number of things! **integrand**: $f(x)$ in the

formula above - the function being integrated

limits of integration: a and b above - they represent the endpoints of the interval over which we are integrating. a is the lower limit and b is the upper limit, due to their position in the formula

$$\int_a^b f(x)dx.$$

infinitesimal: dx is an infinitesimal, having no finite length, but having the same units as x. It represents the infinitely fine width of the approximating rectangle of height $f(x)$, so that $f(x)dx$ is an area (albeit an infinitely small one!). The cool thing is that adding up an infinite number of infinitely small things can give a finite, non-zero answer!

- **Riemann sum**: the sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

named after Bernhard Riemann (1826-1866), a student of Gauss.

- **Midpoint rule**

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i),$$

the midpoint of the i^{th} subinterval $[x_{i-1}, x_i]$.

b. **Theorems**

- $$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

- $$\int_a^a f(x)dx = 0$$

- $$\int_a^b c\ dx = c(b-a)$$

- $$\int_a^b c\ f(x)\ dx = c\ \int_a^b f(x)\ dx$$

- $$\int_a^b [f(x)+g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

- $$\int_a^b [f(x)-g(x)]\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$$

- $$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$$

- If $f(x) \geq 0$ for $a \leq x \leq b$, then
$$\int_a^b f(x)\ dx \geq 0$$

- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then
$$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$$

- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then
$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$$

c. Properties/Tricks/Hints/Etc.

- While we find it convenient to work with subintervals of equal width, there is no need to, and we sometimes use subintervals of varying widths. In that case, the formula for the Riemann sum might be written

  $$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

  and for the definition of the integral we would require that both $n \to \infty$, and $\Delta x_i \to 0$ as $n \to \infty$.

d. Summary Whew! There's a lot going on in this section. The main idea is that we

generalize from area to the integral, which is a way of defining the net area (i.e., some area is considered positive, and some negative; the integral is the sum of both parts for any function). Regions trapped between the curve of f and the x-axis, but above the x-axis, are considered positive in area; those below the x-axis are considered negative.

The integral of a function is a linear operation - that is, if

$$I : f \longrightarrow v$$

is integration (taking a function f and returning a number v), then if $I(f) = v$ and $I(g) = w$ we have that

$$I(af + bg) = av + bw.$$

We are introduced to the midpoint rule, which is an improvement on either the right or left endpoint rules. While subintervals are generally of fixed width Δx, it is not necessarily so, and may sometimes be more convenient to use variable sized rectangles.