Section Summary: 4.3 – The Fundamental Theorem of Calculus

a. Definitions

- **Fresnel function**:
$$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$$
Named after Augustin Fresnel (1788-1827), famous for his work in optics. The Fresnel lens was a miraculous improvement in lighthouse technology: it allowed them to cast a much more powerful beam, and saved many ships and sailors from Davy Jones's locker....

b. Theorems

- **Fundamental Theorem of Calculus, Part I**

  If f is continuous on $[a, b]$, then the function g defined by
$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b$$
  is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

  g is undoubtably one of the oddest functions we have seen so far: it is defined as an integral. Symbolically we might write
$$g : x \longrightarrow \int_a^x f(t)dt$$
  It is a map from the real numbers, represented by variable x, into the real numbers (the values of the integral).

- **Fundamental Theorem of Calculus, Part II**

  If f is continuous on $[a, b]$, then
$$\int_a^b f(x)dx = F(b) - F(a)$$
  where F is any antiderivative of f: that is, a function such that $F' = f$.

  The big news is that if you know an antiderivative of integrand f, then computing an integral is easy.

- **Fundamental Theorem of Calculus**

  Suppose f is continuous on $[a, b]$.

i. If $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$.

ii.

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f.

c. Properties/Tricks/Hints/Etc.

- An important example of this relationship between the function and its antiderivative is given from the physics of integrating velocity:

  $$\int_a^b v(t)dt = s(b) - s(a)$$

  the result is the position function.

- $$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

d. Summary

This section introduces the fundamental theorem of calculus. It contains two parts: it shows that integrals are solved using antiderivatives, and that derivatives of functions defined using variables limits are solved using derivatives:

Suppose f is continuous on $[a, b]$.

i. If $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$.

ii.

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f.

The key is "variable limits": for these functions, the variable is in the **limits** of integration (and **not in the integrand**). One problem, or common reason for misunderstanding, is that there is a "dummy variable of integration" in the problem (no offense!). The variable t in the integral

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

is a **dummy** variable: you notice that t doesn't appear on the right hand side: only x appears, because t has disappeared during integration.