Dakota Langhals
MAT 194

Much Better than my proof Dakota!

Let each square on the board be categorized as a middle (M), side (S), or corner (C) square according to the following diagram:

C	S	C
S	M	S
C	S	C

The properties of all S, M, or C are identical to the other S, M, or C, respectively, because the board is perfectly symmetrical and each square has exactly the same relationship to adjacent squares as its counterpart on the opposite end of the board.

Assume that the goal of each player is to achieve the best result possible. From this, we can identify what I will call "forced moves," which can be defined as the only legal move on a given board arrangement that will avoid defeat. With this assumption, the number of possible games collapses to only a fraction of what it would be if moves were made entirely at random.

great!

A "line" can be defined as any series of moves. Lines that produce forcing moves do not branch out into multiple lines, but only the single continuation which the forcing move dictates.

excellent

There are 3 possible opening moves: M, S, and C. To prove that a draw can always be forced, we need only show that there is one line available to either player which produces a sequence of forcing moves that results in a draw, while there are simultaneously no lines available to the other player which produce a sequence of forcing moves that results in a win.

Let's first consider a scenario in which Player One (P1) chooses to start the game with M. Let's assume Player Two (P2) responds with S.

	X_1	
	O_1	

Immediately, P1 has a forcing sequence which results in their victory. Moving to one of the corners adjacent to O_1 forces P2 to place O_2 in the opposite corner. Now, P1 can simply place X_3 such that he threatens two possible Tic-Tac-Toes. P2 cannot prevent both, so we can conclude that P2 will lose if he responds with S to P1 going M on the first move. Therefore, P2 must play C.

O_2		X_3
	X_1	☆
☆	O_1	X_2

	X_1	
		O_1

Notice that the only squares to which P1 can move without immediately creating a forcing move for P2 on the following move are the two squares adjacent to both X_1 and O_1 and the corner opposite O_1. If P1 chooses to move to none of these squares, he will create a forcing sequence that will either fill the whole board, as in the following example,

X_5	X_2	O_3
O_4	X_1	X_4
X_3	O_2	O_1

or fill most of the board, as in this example,

	O_3	X_2
	X_1	
O_2	X_3	O_1

in which it should be clear that P1 can only make one move that even could win (either the left or right side), but that either one forces P2 to block on the other side and force a draw. Note that this line is identical to placing X_2 in the bottom left corner, and that the previous example is identical to placing X_2 in the left-side square.

What if P1 goes for a win by choosing the squares that do not force P2 to play a forcing move?

X_2		
	X_1	
		O_1

Here, P2 must produce a forcing move or he will allow P1 to achieve the same formation that let P1 win earlier. More specifically, P2 must choose one of the remaining corners (which are identical) or else force P1 to choose the winning formation. Therefore, we get

X_2	O_3	
	X_1	
O_2	X_3	O_1

which is identical to, and is therefore a forced draw. Clearly, P1 cannot force a win by playing into the middle on his first move. What happens if P1 plays S or C on his first move? Knowing that all S and C are identical within their class, consider the following:

	O_1	
	X_1	

Clearly, any X_2, even a forcing X_2, allows an O_2 which will force a particular X_3. Therefore, from just these first two moves, 5 squares will be filled at minimum, as in the following examples, which are exhaustive of all distinct X_2s for P1.

X_3		
O_3	O_1	X_4
X_2	X_1	O_2

[No possible Win]

X_4	O_4	X_3
X_5	O_1	X_2
O_2	X_1	O_3

[Draw]

X_2	O_3	X_3
	O_1	
O_2	X_1	

[Draw]

O_2	X_2	
	O_1	
O_3	X_1	X_3

[Win for P2, but X_2 was not forced]

Because beginning with S as P1 cannot promise a win when O_1 is placed in M, X_1 in S cannot promise a win at all.

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Finally, what if X_1 is placed in a C? Let's assume once more that O_1 is placed in M. There are 4 unique moves P1 can choose in this position: 1 C opposite X_1, 2 S adjacent to X_1, 3 C in top right/bottom left, 4 and S in top/left.

	O_1	
		X_1

Option 1 draws because O_2 will be placed in any S, forcing X_3 to the opposite S, which forces O_3 and X_4 to the remaining corners, leaving only 2 squares, both of which are forced to be filled defensively.

X_2	O_2	X_4
X_5	O_1	O_4
O_3	X_3	X_1

[All forced]

Option 2 draws because O_2 is forced to the C that blocks, which forces X_3 to the opposite corner to block, which forces O_3 to the S that blocks, which forces X_4 to block in the opposite S, leaving O_4 and X_5 to fall uselessly into the last 2 squares

O_4	X_4	O_2
X_5	O_1	X_2
X_3	O_3	X_1

Option 3 draws because O_2 is forced to the S that blocks, forcing X_3 to the opposite S. This only leaves four squares, but P2 forces a draw by playing to either remaining side and following the resulting forced X_4.

X_5	O_3	X_2
X_3	O_1	O_2
O_4	X_4	X_1

Finally, Option 4 draws because O_2 will be placed on any non-opposite S, forcing X_3 to the opposite S. At this point, P1 has no connected Xs but only has 2 more moves, which P2 can easily keep from connecting. Therefore, X_1 being placed on C cannot force a win. Because no opening move can force a win, there is no way to force a win at Tic-Tac-Toe given the assumptions noted at the beginning of this assignment.

X_5	X_2	O_3
O_2	O_1	X_3
X_4	O_4	X_1