

Much Better Than  
my proof  
Dakota!

Let each square on the board be categorized as a middle (M), side (S), or corner (C) square according to the following diagram:



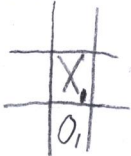
The properties of all S, M, or C are identical to the other S, M, or C, respectively, because the board is perfectly symmetrical and each square has exactly the same relationship to adjacent squares as its counterpart on the opposite end of the board.

Assume that the goal of each player is to achieve the best result possible. From this, we can identify what I will call "forced moves," which can be defined as the only legal move on a given board arrangement that will avoid defeat. With this assumption, the number of possible games collapses to only a fraction of what it would be if moves were made entirely at random.  
great!

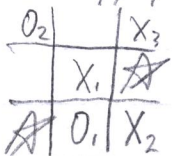
A "line" can be defined as any series of moves. Lines that produce forcing moves do not branch out into multiple lines, but only the single continuation which the forcing move dictates.

There are 3 possible opening moves: M, S, and C. To prove that a draw can always be forced, we need only show that there is one line available to either player which produces a sequence of forcing moves that results in a draw, while there are simultaneously no lines available to the other player which produce a sequence of forcing moves that results in a win.

Let's first consider a scenario in which Player One (P1) chooses to start the game with M. Let's assume Player Two (P2) responds with S.



Immediately, P1 has a forcing sequence which results in their victory. Moving to one of the corners adjacent to O<sub>1</sub> forces P2 to place O<sub>2</sub> in the opposite corner. Now, P1 can simply place X<sub>3</sub> such that he threatens two possible Tic-Tac-Toes. P2 cannot prevent both, so we can conclude that P2 will lose if he responds with S to P1 going M on the first move. Therefore, P2 must play C.



✓

X <sub>1</sub>		
		O <sub>1</sub>

Notice that the only squares to which P1 can move without immediately creating a forcing move for P2 on the following move are the two squares adjacent to both X<sub>1</sub> and O<sub>1</sub> and the corner opposite O<sub>1</sub>. If P1 chooses to move to none of these squares, he will create a forcing sequence that will either fill the whole board, as in the following example,

X <sub>5</sub>	X <sub>2</sub>	O <sub>3</sub>
O <sub>4</sub>	X <sub>1</sub>	X <sub>4</sub>
X <sub>3</sub>	O <sub>2</sub>	O <sub>1</sub>

or fill most of the board, as in this

Example,

	O <sub>3</sub>	X <sub>2</sub>
	X <sub>1</sub>	
O <sub>2</sub>	X <sub>3</sub>	O <sub>1</sub>

in which it should be clear that P1 can only make one move that even could win (either the left or right side), but that either one forces P2 to block on the other side and force a draw. Note that this line is identical to placing X<sub>2</sub> in the bottom left corner, and that the previous example is identical to placing X<sub>2</sub> in the left-side square.

What if P1 goes for a win by choosing the squares that do not force P2 to play a forcing move?

X <sub>2</sub>		
	X <sub>1</sub>	
		O <sub>1</sub>

Here, P2 must produce a forcing move or he will allow P1 to achieve the same formation that let P1 win earlier. More specifically, P2 must choose one of the remaining corners (which are identical) or else force P1 to choose the winning formation. Therefore, we get

X <sub>2</sub>	O <sub>3</sub>	
	X <sub>1</sub>	
O <sub>2</sub>	X <sub>3</sub>	O <sub>1</sub>

which is identical to, and is therefore a forced draw. Clearly, P1 cannot force a win by playing into the middle on his first move. What happens if P1 plays S or C on his first move? Knowing that all S and C are identical within their class, consider the following:

	O <sub>1</sub>	
	X <sub>1</sub>	

Clearly, any X<sub>2</sub>, even a forcing X<sub>2</sub>, allows an O<sub>2</sub>

which will force a particular X<sub>3</sub>. Therefore, from just these first two moves, 5 squares will be filled at minimum, as in the following examples, which are exhaustive of all distinct

X <sub>3</sub>		
O <sub>3</sub>	O <sub>1</sub>	X <sub>4</sub>
X <sub>2</sub>	X <sub>1</sub>	O <sub>2</sub>

X <sub>4</sub>	O <sub>4</sub>	X <sub>3</sub>
X <sub>5</sub>	O <sub>1</sub>	X <sub>2</sub>
O <sub>2</sub>	X <sub>1</sub>	O <sub>3</sub>

X <sub>2</sub>	O <sub>3</sub>	X <sub>3</sub>
	O <sub>1</sub>	
O <sub>2</sub>	X <sub>1</sub>	

O <sub>2</sub>	X <sub>2</sub>	
	O <sub>1</sub>	
O <sub>3</sub>	X <sub>1</sub>	X <sub>3</sub>

X<sub>2</sub>s for P1. Because beginning with S as P1 cannot promise a win when O<sub>1</sub> is placed in M, X<sub>1</sub> in S cannot promise a win at all.

[No possible win]

[Draw]

[Draw]

[Win for P2, but X<sub>2</sub> was not forced]

Finally, what if  $X_1$  is placed in a C? Let's assume once more that  $O_1$  is placed in M. There are 4 unique moves P1 can choose in this position: ① C opposite  $X_1$ , ② S adjacent to  $X_1$ , ③ C in top right/bottom left, and ④ S in top/left.

	O <sub>1</sub>	
		X <sub>1</sub>

Option 1 draws because  $O_2$  will be placed in any S, forcing  $X_3$  to the opposite S, which forces  $O_3$  and  $X_4$  to the remaining corners, leaving only 2 squares, both of which are forced to be filled defensively.

X <sub>2</sub>	O <sub>2</sub>	X <sub>4</sub>
X <sub>5</sub>	O <sub>1</sub>	O <sub>4</sub>
O <sub>3</sub>	X <sub>3</sub>	X <sub>1</sub>

[All forced]

Option 2 draws because  $O_2$  is forced to the C that blocks, which forces  $X_3$  to the opposite corner to block, which forces  $O_3$  to the S that blocks, which forces  $X_4$  to block in the opposite S, leaving  $O_4$  and  $X_5$  to fall uselessly into the last 2 squares

O <sub>4</sub>	X <sub>4</sub>	O <sub>2</sub>
X <sub>5</sub>	O <sub>1</sub>	X <sub>2</sub>
X <sub>3</sub>	O <sub>3</sub>	X <sub>1</sub>

Option 3 draws because  $O_2$  is forced to the S that blocks, forcing  $X_3$  to the opposite S. This only leaves four squares, but P2 forces a draw by playing to either remaining side and following the resulting forced  $X_4$ .

X <sub>5</sub>	O <sub>3</sub>	X <sub>2</sub>
X <sub>3</sub>	O <sub>1</sub>	O <sub>2</sub>
O <sub>4</sub>	X <sub>4</sub>	X <sub>1</sub>

non-opposite

Finally, Option 4 draws because  $O_2$  will be placed on any S, forcing  $X_3$  to the opposite S. At this point, P1 has no connected Xs but only has 2 more moves, which P2 can easily keep from connecting. Therefore,  $X_1$  being placed on C cannot force a win. Because no opening move can force a win, there is no way to force a win at Tic-Tac-Toe given the assumptions noted at the beginning of this assignment.

X <sub>5</sub>	X <sub>2</sub>	O <sub>3</sub>
O <sub>2</sub>	O <sub>1</sub>	X <sub>3</sub>
X <sub>4</sub>	O <sub>4</sub>	X <sub>1</sub>