Mathematical Anthropology
Author(s): Hans Hoffmann
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MATHEMATICAL ANTHROPOLOGY

Hans Hoffmann • *State University of New York at Binghamton*

Although human imagination is unbounded, our unaided ability to experience it is limited. Experiencing requires tools, and as these become developed wider realms of imagination can be made one's own. We can imagine differences in the length of objects, but need a tool—the natural numbers—to experience them. We can imagine an infinity of numbers beyond the integers and their inverses, but need a tool—Cantor's diagonal proof—to experience their existence. For this reason, tools are the essence of culture; whether physical or mental, they permit man to experience wider ranges of his universe. Atlatls permitted early man to experience the flesh of mammoths, rather than the other way round. Biochemistry is a tool that can preserve life, and thus allow more people to experience a full life. Division of labor is a tool that enables man to experience complex societies and urban life. Mathematics is a tool that enables man to understand and control an immense number of events and processes in the physical world.

Mathematics, in particular, is a tool that penetrates realms of imagination hopelessly beyond the experience of a toolless mind. Moreover, once mathematical tools have been developed, they often reverse their effect and enlarge not only one's experience but also one's imagination. Manning (1914: 13) observed:

> The four-dimensional geometry is far more extensive than the three-dimensional, and all the higher geometries are more extensive than the lower. The number and variety of figures increases more and more rapidly as we mount to higher and higher spaces, each space extending in a direction not existing in the lower spaces, each space only one of an infinite number of such spaces in the next higher.

Mathematics, like all of human culture, is constantly growing. Tools —or problem solutions—developed by earlier innovators are compressed, generalized, and stored to become the cultural inventory of today. This is as true of an automobile as it is of geometry. Automobiles combine pneumatic tires, internal combustion engines, and thousands of other innovations to solve the problem of human immobility. Geometry collects the surveying rules of the Egyptians, the deductive procedures of the Greeks, the non-Euclidean formulations of the nineteenth century, and the movement beyond spaces of only three dimensions, and from them constructs a tool for the exploration of space in general.

Cultural solutions are stored in a variety of forms—physical technologies, mythologies, social organizations, and written records (Roberts 1964). Mathematical solutions are usually stored in the form of theorems, each a concise description of certain structural aspects of a mathematical system. For convenience, solutions to mathematical problems are arranged in sequence from the most general to the most restricted, and each successive theorem is derived from the more general ones preceding it. The sequence as a whole thus takes the form of a deductive system. Euclidean geometry is one classic example of such a system; the evolution of topology out of set theory is another.

It is as difficult to define "mathematics" to the satisfaction of all mathematicians as it is to define "culture" to the satisfaction of all anthropologists. However, as Abraham observes (1966: 3–6),

> In mathematics, nothing is more elemental and pervading than the notions of *set* and *mapping*. . . . Much of mathematics is concerned with the structure and behavior of a special class of mappings. In fact, the various branches of mathematics can be described as the study of certain categories, or classes of sets with a certain type of structure, together with mappings which preserve this structure. For example, linear algebra is concerned with the category of vector spaces and linear mappings.

As an illustration, consider the one-to-one mapping that associates an element of the set "integers" with an element of the set "squares of integers" (corresponding to the function $y = x^2$). This particular mapping opens up a whole realm of mathematics, known as transfinite arithmetic. We notice first of all that neither of these two sets is finite; and further, that the set "squares of integers" is a subset of the set "integers." These mathematical observations were first made by Galileo, and led to the modern definition of a (denumerable) infinite set as one that can be put into one-to-one correspondence with a (proper) subset of itself.

Mathematics and science are very different systems, perhaps as different as male and female. Yet their union has proven to be enormously productive. Mathematical analyses of empirical data are the essence (indeed, the criteria) of the hard sciences. In return, empirical problems have stimulated much mathematical innovation. It is difficult to conceive of one discipline existing without the other. Mathematics is concerned with systematic relationships between abstract sets that have no empirical content whatever. The results of this treatment are variously described. Braithwaite, for example, employs the term "calculus" (1953: 23):

A representation of a deductive system in such a way that to each principle of deduction there corresponds a rule of symbolic manipulation will be called a *calculus*. The use of a calculus to represent a deductive system has the enormous practical advantage that it enables deductions to be effected merely by symbolic manipulation, and the correctness of these deductions can be checked automatically merely by inspecting the relationship between the symbols; it is for this reason that the Indian invention of Arabic numerals was such a landmark in the history of civilization.

Geoghegan prefers the term "axiomatic theories," which he describes as follows (1965: 6):

This approach to theory construction usually involves four basic elements: (1) a specification of primitive notions, (2) a statement of the axioms, (3) a presentation of relevant definitions, and (4) the derivation of useful theorems. . . . Discussion of the primitive notions usually constitutes part of the theory's interpretation: the assignment of "meaning" to otherwise "meaningless" logical entities.

Science, by contrast, is concerned with systematic relationships between sets of concrete empirical data.

The function of science . . . is to establish general laws covering the behavior of the empirical events or objects with which the science in question is concerned . . . and to make reliable predictions of events as yet unknown. If the science is in a highly developed stage, as in physics, the laws which have been established will form a hierarchy in which many special laws appear as logical consequences of a small number of highly general laws expressed in a very sophisticated manner; if the science is in an early stage of development—what is sometimes called its "natural history" stage—the laws may be merely the generalizations involved in classifying things into various classes. (Braithwaite 1953: 1.)

Clearly, both mathematics and science are organized as deductive systems. The difference between them depends on whether the sets treated by the systems are abstract or concrete.

A laboratory scientist or field worker usually employs some abstract deductive system developed by mathematicians when he is constructing a scientific system, since he rarely possesses the specialized training to construct an effective mathematical system of his own. To expect an anthropologist, for example, to invent and develop Markov chains from scratch in order to investigate Ethiopian age grades (Hoffmann 1965) is asking for rather a lot. In this way, I availed myself of Fitch's mathematical system of symbolic logic (1952) in developing a scientific theory of Pawnee marriage rules (1959). Geoghegan, on the other hand, acted as both mathematician and scientist in constructing his theory of information processing (1965: 7–8).

Even though a theory and its interpretation are completely different things, it is often the case that they are presented simultaneously, usually through a judicious choice of familiar terminology. . . . By constructing an economical axiomatic basis [mathematical system] for the [scientific] theory, we can minimize the amount of interpretation required to make it meaningful, and thereby eliminate many of the semantic problems that might otherwise confront us in using the theory and constructing productive models for it.

Braithwaite's criterion for a developed science—"a hierarchy in which many special laws appear as logical consequences of a small number of highly general laws"—is not often met in contemporary anthropology. My own hierarchy of eleven theorems (1959) that generated the Pawnee marriage rule was an early move in this direction. The ten theorems that constitute Geoghegan's theory of information processing are a much tighter hierarchy, and may well be a model for future anthropological theories. Neither of these systems has a wide application: mine applies to only one culture type, and Geoghegan's to only a small part of any given culture (unless one takes the view that all culture is merely information-processing activity). However, these are essential beginnings.

Randall has called attention to a fundamental division among systems, be they scientific or mathematical. Scientific systems associate an input (e.g., a quantity of heat) with a definite output (the temperature of a liquid). The relationship between input and output depends on the properties of the system (here, the specific heat and the volume of the liquid). Randall continues (1968: 24–25):

With Nering (1963: 1–2), let us suppose that the system transforms its inputs in certain ways. Assume that two inputs can be added and their sum put through the systems so as to produce an output. Also assume that any two separate inputs will produce two separate outputs. If the sum of those outputs is equal to the output of the sum, then the system is known as an *additive system*. Also assume that an input can be changed by a constant-factor multiplication. In an additive system, if the output is changed by a particular factor, then the system is called *linear*. Determinant, Markovian stochastic, differential, and integral operators are all linear systems.

Of course, not all empirical systems are linear. . . . For this reason, some nonlinear systems of classical physics have been approximated as linear in order to achieve solution. For example, the restoring force of a simple pendulum is proportional to the sine of the vertex angle. Even though trigonometric functions are not linear, a good model can be made (subject to the constraint that the angle is small) by approximating the sine of the angle as the angle itself (Sears and Zemansky 1955: 201). Also, a few human physiological systems have been successfully modeled by such approximations (Grodins 1963: 27).

In short, elegant mathematical solutions to nonlinear problems are rare or nonexistent (Nering 1963: 1). Hence the behavior of almost all anthropological components must be modeled by linear systems if they are to be modeled at all.

Nonlinear systems are equally common, and equally intractable, in pure mathematics. Mathematicians cope with them in the same way that physicists do: they approximate them with linear systems. Abraham (1966: 37–38) comments:

Linear mappings are of central importance in advanced calculus, which is the study of general nonlinear mappings from one vector space to another. In that context, the derivative $f'(v)$ of a mapping $f\colon V \rightarrow W$, generalizing the familiar tangent line interpretation of the derivative $g'(x)$ of a function $g\colon R \rightarrow R$, is defined to be the linear mapping $L\colon V \rightarrow W$, which is the best possible approximation to f in the vicinity of the point

$v \in V$. (See Spivak, *Calculus on manifolds.* New York, Benjamin, 1965.) Thus nonlinear mappings are studied by approximating them with linear mappings, which have a simpler structure. This idea is the basis of a wide variety of applications of linear algebra to the real world. For example, a factory is a nonlinear mapping from the mathematical viewpoint.

These properties of linear and nonlinear systems explain the practical difficulties faced by generative grammarians. As Randall (1968: 39) points out,

Grammatical space is not linear, and apparently cannot be modeled as linear. More than a decade ago, Chomsky showed that English grammar cannot be modeled by any Markovian (linear) operators when the phase space is assumed to be some finite structured set of connected symbols (Chomsky 1965: 111). Furthermore, it appears likely that these "finite state grammars" are unable to model most natural languages (Chomsky and Miller 1965: 157). Languages usually have a very small number of symbolic inputs and an infinity of outputs that cannot be generated by linear operators.

It is likely that the empirical processes underlying language are very different from those underlying culture, if only because linear models do apply to cultural phenomena. On the other hand, I do not feel qualified to discuss the history and range of generative grammars, and will not deal further with them in this survey. For the same reason, and with considerable regret, Buchler and Selby's *A Study of myth,* based largely on the methods of generative grammarians, will not be treated here. This paper stresses the use of mathematical systems in defining fundamental anthropological concepts and processes, and does not deal with studies that are primarily inductive in emphasis. Therefore, Driver and Schuessler's correlational analysis (1967) of Murdock's 1957 ethnographic sample, the Binfords' analysis of functional variability in the Mousterian of Levallois facies (1966), and similar studies, have also been excluded.

This paper will survey mathematical systems of increasing complexity—partial order, total order, natural numbers, real numbers, vector spaces, and matrices—and consider their application to anthropological problems. I have also included a note on the mathematics of kin-term functions, prepared for this review by Robert Randall. This note illustrates how another taxonomy of algebraic structures can be used to evaluate statements in culture theory. Although there is as yet no literature on this approach, its potential is clearly enormous.

THE DATA SET X

Anthropology, in essence, deals with *sets*—sets of people, events, artifacts, emotions, kinship terms, and many other empirical phenomena. Anthropology is perhaps unique among the scientific disciplines in the extraordinary variety of its data. It is more than just the passive contemplation of data, however, even though many anthropologists are drawn to their field by an intuitive and emotional involvement with an exotic people. Anthropology is a science in that it investigates the actual structure of events, kinship terms, and the like; but it is also an understanding of people and their life that can be communicated to others. If for no other reason, an anthropologist's emotional involvement with his data must be made explicit. Mathematics is the most efficient instrument yet developed to explain structures of all kinds, and is thus the natural language for bringing a private experience to a world of listeners. Mathematics does not destroy intuition; rather, it communicates intuition in a comprehensible form.

THE PRODUCT SET X^2

One of the most fundamental concepts of mathematical structure is that of a *product set*, or Cartesian product. Let X and Y be two sets. The product set $X \times Y$ consists of ordered pairs (x, y) in which x is an element of X, and y of Y. The product of a set with itself, $X \times X$, will be denoted by X^2. Consider X as a set of people that forms a tribe. The product set X^2 will then include all possible pairs of tribesmen. If there are ten people in X, then X^2 will consist of 10^2 elements. A product set of tribesmen can always be defined empirically, even though not all its elements are of anthropological interest. For example, some pairs of tribesmen are related to one another, and others are not. Students of kinship are primarily interested in the first, rather than the second, of these two subsets of X^2.

By contrast, a product set of events, rather than things or people, cannot always be defined empirically in its entirety. Consider the set of events that make up the *fiesta de matrimonio* among the Tenejapa ladino (Metzger and Williams 1963). Some of the ordered pairs of events will reflect Tenejapa reality, but many will not. *Presentación* precedes and is adjacent to *despedida,* but not to *casamiento*. Thus the element (*presentación, despedida*) reflects ethnographic reality, whereas the element (*presentación, casamiento*) does not. It frequently happens that a mathematical abstraction goes far beyond empirical reality, and must be "pruned" before it becomes useful in a science. (Linear programming exploits this procedure, in that the pruning is done by intersecting half-planes. The intersections then become crucial to the solution of the programming problem.) Brewer (1966) has used unpruned product sets in abstracting statements of anthropological theory. However, the concept usually becomes most useful to scientists when they have some means of partitioning off meaningful subsets.

RELATION: R_i

The mathematical entity that partitions a subset out of a product set is called a *relation.* A relation R_i from a set X to a set Y assigns to each pair (x, y) in $X \times Y$ exactly one of the following statements: "x is related to y," or "x is not related to y." A relation from X to X is called a relation in X. In the Tenejapa example, the relation involved would be called "is adjacent to and precedes." This relation can then abstract an empirical part of the structure of Tenejapa wedding ceremonials. Relations have sometimes been explicitly used in the analysis of social organization. For example, among the Pawnee (Hoffmann 1959), certain pairs of tribesmen stand in the relationship *tiwatsiriks* to one another. Empirically, this means that those Pawnee (x) who are called *tiwatsiriks* by other Pawnee (y) are male same-generation agnatic kinsmen of ascending-generation uterine kinsmen of y. Mathematically, this ethnographic observation is abstracted by

$$(x)\Big[(\exists y)[xR_1y] \supset x[[M \cap G^0 \cap A] \mid [G^+ \cap U]]y\Big].$$

Obviously, not all elements of the product set of Pawnee will stand in this relationship to one another. Thus the relation R_1 partitions out one of this set's anthropologically meaningful subsets.

The use of a relation to split a product set imposes a *constraint* on the product set. Ashby (1956: 127, 130) defines a constraint as a relation between two sets, occurring "when the variety that exists under one condition is less than the variety that exists under another."

It follows that *every law of nature is a constraint*. Thus the Newtonian law says that, of the vectors of planetary positions and velocities which might occur, e.g., written on paper [the larger product set], only a smaller set will actually occur in the heavens; and the law specifies what values the elements will have. From our point of view, what is important is that the law *excludes* many positions and velocities, predicting that they will never be found to occur.

Thus R_1 is a constraint inherent in Pawnee culture. It excludes those pairs of Pawnee who cannot call each other *tiwatsiriks* because they are not linked by the genealogical chain that is specified by the definition of R_1. Other relationships in Pawnee culture give anthropological significance to other subsets of elements in the product set. Moreover, since any given Pawnee is involved in more than one kinship relation, these subsets will overlap. The superset of anthropologically meaningful subsets of the Pawnee product set, although it is an abstraction, will not be disjoint.

We should note in passing that kinship relations are not equivalence relations. That is, they are not reflexive (xR_1x) because a father (x) cannot be a father to himself. They are not symmetric (xR_1y and yR_1x) because the son (y) of a father (x) cannot also be the father of his father. Finally, they are not transitive (xR_1y and yR_1z imply that xR_1z) because the father (x) of a son (y) cannot also be the sociological father of that son's son (z). However, a kinship relation does define a *topology* on the product set X^2 of pairs of tribal members (Hoffmann 1968: 50). A *topology* (T) is a class of subsets of the set X^2 if it fulfills three conditions: (1) if the sets X^2 (all the pairs) and $\emptyset$ (none of the pairs) are both included in T; if (2) the union of any number of the subsets in T belongs to T (here, the pairs that are R_1-related together with the pairs that are not R_1-related constitute X^2); and (3) the intersection of any two subsets in T belongs to T (here, the pairs that are R_1-related and not R_1-related constitute $\emptyset$). The mathematical system (X^2, T) is called a *topological space.*

Buchler and Selby (1968a: 279–309) have implicitly used relations to investigate information theory and social organization.

When the observer maps Kariera kin classes onto sections, the information necessary to define the marriage status of an individual is reduced to two bits of information. For example,

1. Is it Banaka or Karimera? (No)
2. Is it Palyeri? (Yes)

Here, each question asked of an informant establishes a relation. In this case, the two relations narrow down the relevant subsets until the ethnographer has established the marriage status of an individual.

PARTIAL ORDER: $\precsim$

In general, the anthropologically relevant subset of elements of a product set is further structured by the culture involved. This may be abstracted as a partial order, defined by a culture, on the subset. Consider the following sequences of events that enable a land unit to move down the generations without being divided in the process (Prindle 1967: 20). The culture is Tibetan.

Marriage in Ladak is either *bag-ma* (patrilineal and patrilocal) or *mag-pa* (matrilineal and matrilocal). Prince Peter believes that *bag-ma* marriage is a fraternally polyandrous one in most cases and all the husband's brothers become the *de facto* husbands of the bride. But, ordinarily, only the eldest son and the next eldest participate in the marriage ceremony (Prince Peter 1963: 346). A *mag-pa* marriage can be monogamous or polyandrous, whichever the heiress decides. If the *mag-pa* marriage is polyandrous, it is usually nonfraternal and the woman often chooses husbands who are not related (Prince Peter 1963: 346).

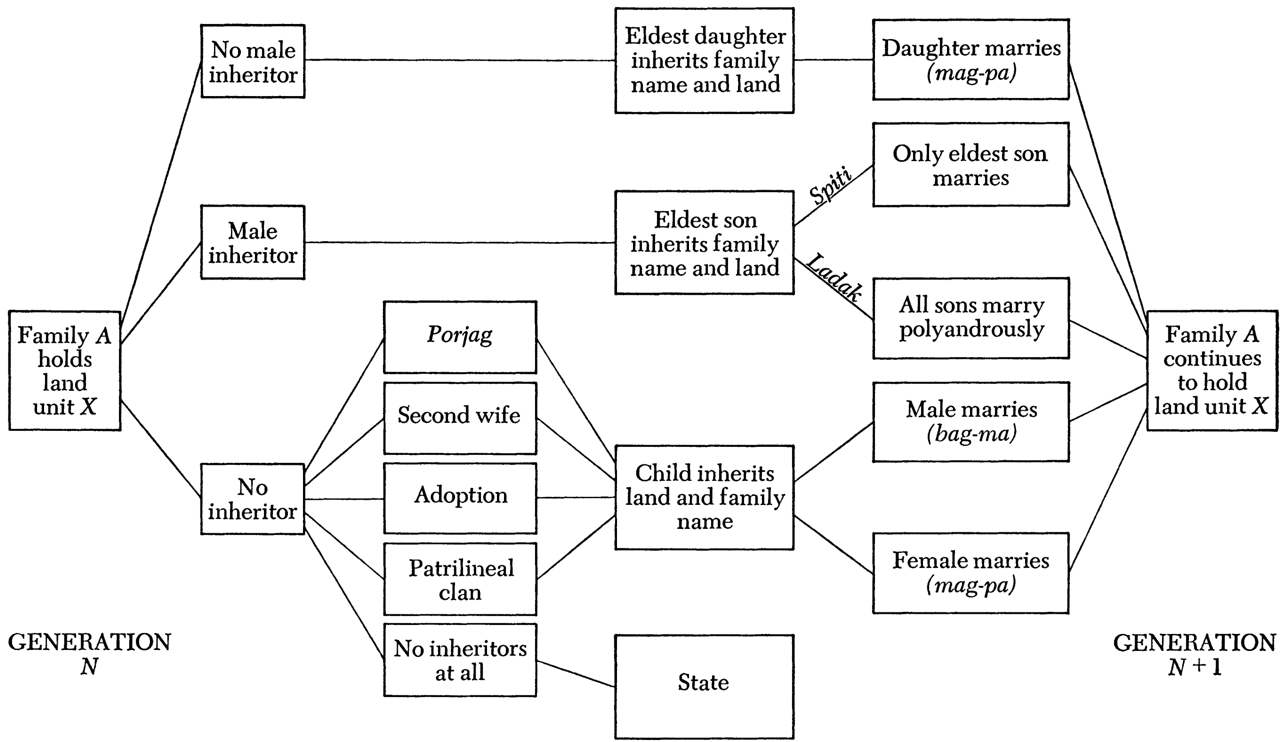
In case a family is childless, every effort possible is employed to provide an heir. If the first wife is barren, the husbands may marry a second or third wife in an effort to provide an heir (Prince Peter 1963: 346). In case a family is childless and not rich enough to bring in another wife, the custom of *porjag* may be employed. By means of this custom, a man is temporarily brought into the household to sire a child (Prince Peter 1963: 346).

Prindle (1967) has isolated 19 events (enclosed in boxes in the diagram), which would form a product set of 19^2 (or 361) elements. Only 25 of these elements (culturally permitted transitions between events) are actually observed in Tibetan culture. (See the chart on p.

52; modified from Prindle 1967.) These elements form a partial order of the product set. A relation $\precsim$ in a set Y is called a *partial order* on Y if it is reflexive ($a \precsim a$), antisymmetric ($a \precsim b$ and $b \precsim a$ imply that $a = b$), and transitive ($a \precsim b$ and $b \precsim c$ imply that $a \precsim c$). The mathematical system $(Y, \precsim)$ is called a *partially ordered set.* Culturally permitted transitions between ethnographic events are a partially ordered set with $\precsim$ defined as "event a can follow event b in time."

Another variety of order can be defined in a set, with vast consequences for both mathematics and science. This relation is called total (or linear) order, and is symbolized $<$. A set is totally ordered if the relation $\precsim$ can be defined for every pair of its elements, that is, if either $a \precsim b$ or $b \precsim a$ holds for every pair of elements (a, b) in the set. The mathematical system $(X, <)$ is called a *totally ordered set.* Some subsets of a partially ordered set Y may in fact be totally ordered. Although it is not possible to define the relation $\precsim$ for the pair of elements "porjag" and "male inheritor," there are paths through Prindle's diagram where any two events encountered can be so ordered. One example would be: family A holds land, no inheritor, porjag, child inherits, female marries mag-pa, family A continues to hold land. Culture patterns may be thought of as totally ordered sets that form a partially ordered superset.

Partially ordered sets can also be discussed with the language of graph theory. Graphs consist of vertices (events), edges (transitions between events), and a mapping that associates each edge with a pair of vertices (Busacker and Saaty 1965: 6). In other words, elements of a product set of vertices are mapped into a set of edges. This mapping has the form: $X^2 \rightarrow Y$. Anthropologists postulate that culture is never static. For example, some tribal culture patterns may be simplified by the tribesmen themselves. A graph of the new pattern would have fewer vertices and fewer edges. Atkins and Curtis (1966) have discussed the metrical aspects of such "de-constraints," using Metzger and Williams's 1963 analysis of Tenejapa ladino weddings. Specifically, Atkins and Curtis propose a "de-constraint" index based roughly on the ratio between number of edges found in a graph before and after the simplification of the culture pattern.



Family A holds land unit X
No male inheritor
Male inheritor
No inheritor
Porjag
Second wife
Adoption
Patrilineal clan
No inheritors at all
Eldest daughter inherits family name and land
Eldest son inherits family name and land
Child inherits land and family name
State
Spiti
Ladak
Daughter marries (mag-pa)
Only eldest son marries
All sons marry polyandrously
Male marries (bag-ma)
Female marries (mag-pa)
Family A continues to hold land unit X
GENERATION N
GENERATION N + 1

Every descriptive ethnography is an extensive graph in latent form. Explicit use of graph theory, however, is a more recent phenomenon. Buchler and Selby (1968a: 48–65) have investigated myths from a graph-theory point of view. After considering Lévi-Strauss's (1963: 227) ordering of the variants of the Hopi Shalako myths, they establish such theorems as: "All versions of a myth may be logically and sequentially derived from an initial transformation relationship or from the composite graph of a group of variants." Busacker and Saaty remark (1965: 4) that every anthropological graph of interest is abstractly identical to a geometric graph, i.e., a curve in E^n. This mathematical entity will be discussed later in the paper. For the moment, we call attention to the implication of their remark, which suggests that all ethnographic data can be converted into numbers.

TOTAL ORDER: $<$

The goal of much anthropological research is to define a total order relation on a set of ethnographic data. Carneiro (1968) has attempted to order tribal groups throughout the world with the relation "tribe *a* has more culture traits than tribe *b*." He is also interested in ordering selected culture traits in Anglo-Saxon England into a developmental hierarchy; here the relation is "trait *a* was developed before trait *b*." The mechanical procedure used by Carneiro to generate these totally ordered sets is Guttman scaling. Buchler (1964) has used the same method to construct a developmental typology of Crow kinship systems. Kay (1964) has used Guttman scaling to order Tahitian consumer behavior, and has discovered such domestic ideals as "stoves outrank refrigerators." More recently, Buchler (1967a) has investigated the Atempan religious hierarchy, which is made up of offices (cargos) devoted to the performance of church ritual. His aim is to uncover the relative weights of the different criteria used in the selection of cargos in order to probe decision-making processes in Atempan. Buchler generated this order of qualifications by using integral linear programming. This is a self-contained computational device, which is essentially independent of the general theory of linear programming based on polytopes in E^n (Gale 1960: 132–79).

THE SYSTEM N

Mathematical systems are made up of elements and an associated structure consisting of the algebraic operations, or mappings, that can be performed in the system. Consider the subset N of the rational integers $Z = (0, \pm 1, \pm 2, \pm 3, \ldots)$. The elements in N are called the natural numbers, and consist of $(1, 2, 3, \ldots)$. The system N contains the operation of addition, defined as a mapping, α, that takes a Cartesian product of elements from N back into N, that is,

$$\alpha : N \times N \rightarrow N : (m, n) \rightarrow m + n .$$

It also contains the operation of multiplication, β, defined by

$$\beta : N \times N \rightarrow N : (m, n) \rightarrow mn .$$

A third property of the natural numbers is that the total order relation $<$ can be defined on them. We are now in a position to define the mathematical system N as follows: $N = (N, \alpha, \beta, <)$.

MAGNITUDE: $X \rightarrow N$

A scientific discourse that includes numbers in its vocabulary has enormous power and clarity, compared to one that does not, because it can deal with the concept of magnitude in very precise terms. But any scientific vocabulary that includes numbers must consist of an interpreted mathematical system. The statement "this village contains ten houses" implies that elements of a *discrete* set, that of houses in a given village, have been paired off with elements of the mathematical system N, which includes the natural numbers (i.e., the non-negative real integers). The magnitude of the set of houses is established by the cardinality of the subset of N needed for the mapping. If the last house is mapped into the tenth element of this subset, the cardinality of the set of houses is ten.

ADDITIVITY: $X + Y \rightarrow N$

It is possible to define the concept "magnitude of two discrete sets" using only the system N. This application of N to an empirical problem is made possible by the existence of the mapping α within N. The magnitude of two discrete sets taken together is the sum of the magnitudes of the individual sets. A village with ten houses in one part and five in another contains fifteen houses altogether. This, of course, is how we expect the concept of magnitude (M) to behave. Abstractly, this expectation is reflected in the *additivity* of magnitude: $M(A+B) = M(A) + M(B)$. Additivity can be taken on the level of postulate. It is a characteristic of mathematics that seeming trivia of this kind are vitally important. For example, from this postulate Lebesgue developed in 1902 a measure of the magnitude of absolutely discontinuous point sets. "Lebesgue's discovery came none too soon, for already discontinuity had begun to invade physics.... Now it began to be realized that the structure of electricity, matter, and energy was granular, so that measures of these quantities varied in jumps, or discontinuously" (Singh 1959: 127).

REAL NUMBERS: R

Frequently, scientific investigations require that numbers be associated with sets of continuous rather than discrete empirical data. This can produce complications that the system N is unable to resolve. Suppose that the length of a house is to be measured by repeatedly placing a standard yardstick against the house. One can then count how many times the stick was used; thus the magnitude of a continuous set has been established by counting the elements of a discrete set. So far, so good. But what if the house is more than 5 and less than 6 yards long? There is no way to measure the remainder with an unmarked yardstick. More generally, there is no element $\frac{1}{2}$ in the set N. To establish magnitudes of sets of continuous empirical data, which generally contain remainders, a different, augmented, mathematical system is required. Many empirical contingencies like this embarrass the system N. One cannot include half a pot in an analysis because the rationals like $\frac{1}{2}$ are not elements of N. One cannot measure the diagonal of a square blanket because the irrationals like $\sqrt{2}$ are not elements of N. One is at a total loss to measure the area of a circular field because the transcendentals like π are not elements of N.

Clearly, a mathematical system that includes all of the above number types, as well as the set of natural numbers, is needed. These elements make up the set of *real numbers*, or R. A geometric representation of R includes all of the points on a straight line, not just the points corresponding to integers. The ordering structure associated with the set R is called a *field*, and is defined by a set of postulates omitted here. The real number system R contains the mappings α and β, as well as the order relation $<$. Thus the real number system is defined by $R = (R, \alpha, \beta, <)$.

The real number system is so rich in structure that it supports most of the mathematical systems so far used by scientists. To be sure, "There are a number of seemingly unrelated problems from physics, communication engineering, statistics, and so on that lead us to consider probabilistic relations in algebraic structures not equivalent to the real line" (Grenander 1963: 14). The intellectual charm of this esoterica cannot, as yet, compare with the solid payoff of R-based systems. R is required if half-finished houses are to be counted, and anthropological variables must be able to range over all of the real numbers if mathematical anthropology is to have much substance. The situation is not desperate. Sociological subjects produce 2.3 offspring with bemused indifference to mathematical impropriety; compulsive potters hand unfinished wares to apprentices, who may never exceed $\sqrt{2}$ of a design lest the spirits object. Nevertheless, the venture moves, at least for all but the most puritan of mathematicians.

CONTINUOUS SET MAPPING: $X \to R$

Many parameters of human populations are recorded by real numbers: height, weight, IQ, blood pressure, and opinions, for example. These data form continuous sets, whose magnitudes are established with the mapping $X \to R$. The mapping corresponds to a pointer reading, i.e., to a pointer that has been moved by the empirical phenomenon along a scale inscribed with all the real numbers (exactly for the rationals, approximately for the irrationals). Scientific reasoning is based on various abstract concepts such as mass, length, time, temperature, people, or culture traits. These are mutually independent, and have unique units of measurement: gram, centimeter, second, degree, number of people, and number of traits. Each of these units conforms to the postulates for a *measure*, which mathematically abstract the scientific requirement that the "length" of a broken and reassembled stick is equal to the sum of the "lengths" of its parts. This concept has already been introduced in discussing the magnitude of a discrete set. Mathematically, measures are an important subset of the collection of mappings of the form $X \rightarrow R$; they are set functions, i.e., functions (or mappings) that associate a set X with the real numbers.

The postulates for a measure are: $M(0) = 0$ (the measure of the empty set is zero); $M(A) \geqslant 0$ (the measure of a non-empty set is positive); and $M(A) + M(B) = M(A + B)$ (measures are additive).

Since empirical units are the building blocks of scientific discourses, continued attention to their definition and refinement is essential. Naroll (1964) has reopened this quest in anthropology, and thus set the stage for such symposia as *Essays on the problem of tribe* (Helm 1967). However, a great deal of further work is required before the appropriate units of anthropological discourses can be expected to emerge. Carneiro (1957: 169–70) has proposed an index of subsistence productivity that might serve as an anthropological unit—i.e., as a basic concept expressed numerically and conforming to the measure postulates. This index records the number of man-hours required to obtain a specified annual caloric consumption of food (10^6, an average figure for most human populations) from a given mode of subsistence. Carneiro's proposed measure consists of the function $M(P$ of S_i/year $= 10^6/C_i$, where C_i is the number of calories produced per man-hour per mode of subsistence (S_i). Now, $10^6/C_i$ is a real number, so that Carneiro's measure maps each mode of subsistence S_i into the reals by means of the rule: "Take the value of C_i associated with S_i and divide it into a million."

The set of measures contains an important subset, which is defined by postulating that the measures of all its subsets must add up to 1:

$$M(A_1) + M(A_2) + \ldots + M(A_n) = 1 .$$

This postulate divides the *probability measures* from more general measures. This interpretation of the concept of probability was developed by Kolmogorov in 1933, and represents a major breakthrough in developing a language for science: "This task would have been a rather hopeless one before the introduction of Lebesgue's theories of measure and integration. However, after Lebesgue's publication of his investigations, the analogies between measure of a set and probability of an event . . . became apparent" (Kolmogorov 1950: 1). Anthropological applications of probability theory are extensive. An interesting recent example is Goldberg's investigation of FBD marriage among Tripolitanian Jews in Israel (1967: 176–91), which developed a probability measure as follows:

$$\text{Probability of FBD marriage at a given age} = \frac{\text{rate of marriage} \times \text{probability of having available FBD} \times \text{number of youths marrying at a given age}}{\text{number of women in given age range} \times \text{number of males}}$$

This associates the set of bachelors who will marry their FBD with the real number 0.008.

THE MAPPING $R \to R$

Scientific reasoning uses concepts constructed from various basic entities. "Velocity" is associated with a specific number of distance units traversed during a given number of time units ($V = D/T$). Similarly, "acceleration" is defined by $(D/T)/T$, "force" by $M(D/T)T$, "sedentariness" by $PT/(D+1)$, and so on. Mathematically, these concepts are all mappings of the form $R \to R$. Velocity associates a real number V with a real number represented by the ratio D/T. The graph of this function is a subset of the plane R^2, in this case a straight line through the origin (or zero vector) of R^2. Mappings of the form $R \to R$ contain a subset whose members are particularly well suited for scientific interpretation. These are the *injective* or *one-to-one* mappings, defined by: if $a = b$ and $a = c$, then $b = c$. Only one value of V can correspond to a given ratio D/T; hence the mapping interpreted as the definition of velocity must be one-to-one. The source of this imperative is scientific rather than mathematical. There is no reason why functions must be one-to-one on mathematical grounds alone. Mathematically, the falling-bodies function $D = \frac{1}{2}gt^2$ can be one-to-many because D is the same for $t = +1$ and $t = -1$. But negative numbers of time units do not make any physical sense; stones cannot fall for minus ten seconds. Therefore, scientists interpret only that part of the function where $t \geqslant 0$.

As Spaulding (1960: 437–56) has pointed out, there is a second imperative that restricts the interpretation of mappings. Empirical entities have *dimensionality*. Length has dimensionality L, area has L^2, acceleration has LT^{-2}, etc. The dimensionality of both sides of a function must be the same if the function is to make any empirical sense. "Nine cubic feet equal one square yard" is a meaningless definition because the two sides are different ($L^3 \neq L^2$); "27 cubic feet equal one cubic yard," on the other hand, does make sense, since its dimensionality is homogeneous ($L^3 = L^3$). Carneiro (1967: 234–43) has published a function ($N = 0.6P^{0.594}$) intended to abstract his empirical observation that the number of organizational traits in a single-community society is roughly equal to the square root of its population. The dimensionality of this function is $N = P^{\frac{1}{2}}$, which is not homogeneous. It could be recast by taking logarithms, which are dimensionless numbers, and saying that $\log N = 0.594 \log 0.6P$. This would be a dimensionally homogeneous statement of a law of cultural evolution. Or one could define a new variable, "sociocultural complexity," with the ratio $0.6P^{0.594}/N$. The dimensionality of this scientific entity would be $P^{\frac{1}{2}}N^{-1}$. This variable might be quite helpful in future studies of cultural evolution.

THE SET R^n

Frequently, an empirical event in science requires more than one number for its description. Consider a tribal economy involving hunting, fishing, farming, and trading. Its organization can be described by listing the time allocations devoted to each of these pursuits, i.e., by the 4-tuple (x_1, x_2, x_3, x_4). The subscripts refer to a specific economic pursuit, and the x's to the average number of hours per week each family spent in that pursuit. From a scientific point of view, each component of this 4-tuple measures temporal commitment to one economic pursuit, and the 4-tuple as a whole describes a particular organization of tribal culture. From a mathematical point of view, this 4-tuple can be interpreted as one element of a four-fold Cartesian product of the set R with itself, i.e., one element of the set

R^4. This set is defined by $R^4 = R \times R \times R \times R$. Geometrically, this 4-tuple represents one point of a four-dimensional space, and the set R^4 represents all the points of this space. These concepts generalize immediately. The set R^n is the set of all ordered n-tuples of real numbers, i.e., an n-fold Cartesian product of the set R with itself. It is called *Cartesian n-space*, or simply R^n. Geometrically, R^n is an n-dimensional space. Algebraically, R^n is the canonical example of a finite-dimensional real-vector space (Abraham 1966: 21).

Within a vector space, the mapping α is essentially preserved, except that two additional vectors are obtained by adding corresponding components:

$$\alpha : R^n \times R^n \rightarrow R^n : [(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)] \rightarrow (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

This mapping permits time allocations for two different weeks to be added together in a vector space. The mapping β is somewhat different here, and is called *scalar multiplication* (this must be kept distinct from "scalar product," to be discussed later). *Scalar multiplication* is defined by the mapping

$$\beta : R \times R^n \rightarrow R^n : [a, (y_1, y_2, \ldots, y_n)] \rightarrow (ay_1, ay_2, \ldots, ay_n).$$

This mapping permits all time allocations of economic activities to be doubled, should the tribe suddenly work twice as hard. In other words, the components of a vector can be multiplied by any real number once scalar multiplication has been postulated. A vector space, then, is defined by the triple $R^n = (R^n, \alpha, \beta)$.

It must be pointed out that there exist vector spaces other than R^n. Any set satisfying the postulates of a field can supply components for a vector space. One example is the set $(0, 1)$, with the mapping α defined by $0 + 0 = 0$, $0 + 1 = 1$, $1 + 1 = 0$. Thus the vector space R^n is just one member of the category "vector space V over field F." Abraham (1966: 25) points out that in many physical and social science applications of mathematics we encounter vector spaces for which the underlying set is not R^n. A variety of sets replace the n-tuples used in R^n. These more general spaces are called *abstract* or *real-vector spaces*, instead of Cartesian space. Although this point is likely to be of critical importance to the future of mathematical anthropology, we are not in a position to develop it further here.

A good deal of anthropological data can be abstracted as elements of a vector space. Alternatively, the mathematical concept of vector can be interpreted empirically in many ways. In archaeology, for example, seriation diagrams are nothing more than stacks of vectors; each mathematical component measures the relative frequency of each pottery type found at a site. In analyses of social organization, vectors can describe the number of people in different social groups. For example, among the Shoa Galla of Ethiopia, vectors record the percentages of people in each age grade at a point in time (Hoffmann 1965). Buchler and Selby (1968b: 63–67) have used the same procedures in their analysis of Freeman's study of the Iban family system in Borneo (1962). In my own analysis of the economic organization of tribes, vectors were used to describe the time allocations of an upper Amazonian village (Hoffmann 1966: 11–15).

THE MAPPING $R^n \rightarrow R$

In science it is often necessary to associate a real number with a vector. We have already discussed the association of real numbers with one another, i.e., mappings of the form $R \rightarrow R$. We will now consider the anthropological significance of mappings of the form $R^n \rightarrow R$. Although there are an unlimited number of these mappings, the subset partitioned off by the postulates for a norm is of particular interest to scientists. A *norm* associates a real number $||v||$ with a vector v under the following postulates: $||v|| = 0$ (the norm of the zero vector is zero); $||v|| \geqslant 0$ (the norm of a nonzero vector is positive); $||v + w|| \leqslant ||v|| + ||w||$ (the norm of the sum of two vectors is less than or equal to the sum of the norms of these vectors); and $||kv|| = k||v||$ (a norm can be multiplied by a real number). A vector space in which a norm can be computed for each vector is called a normed linear vector space, or simply a *normed space.*

One of the first anthropological interpretations of a norm was made by Randall. Although his entire argument (1968: 73–75) is too complex to review here, it deals with the analysis of adaptation.

> Various changes in social and cultural systems require an adaptive endocrine response in each human witness. I have suggested that these systems can be usefully connected so as to create a general human adaptation system. . . . Moreover, adaptation is a variable that depends on the environ-

ment (i.e., the human systems output vector). Hence I will call the variable "the instantaneous biological adaptation of a human system." . . . The evidence suggests that with appropriate sampling techniques, a biochemical measure of community stress could be obtained for any particular time and population. Furthermore, physiological growth and natural increase rates appear to entail no serious obstacles to measurement. Thus, measuring instruments exist for the establishment of an adaptation norm on a linear model of the human adaptation system.

THE MAPPING $R^n \times R^n \rightarrow R$

Frequently, scientific investigations require that a number be associated with a pair of elements in a vector space. "What is the distance, or length, between the beginning and the end of an arrowhead?" This question sounds trivial, but the mathematical system "vector space over R" is unable to deal with it. Vector spaces have one important limitation: their elements do not constitute a totally ordered set. Although it is plausible enough to call the element $(2, 3, 5)$ "smaller" than the element $(4, 6, 10)$, how would one rank $(8, 1, 3)$ vis-à-vis $(2, 10, 7)$? In fact, there is no way to do so. A primary reason for introducing numbers into science is to compare the magnitude of different quantities. In vector spaces, however, we cannot even tell which of two "numbers" is larger than the other, let alone specify how much larger. What is missing is the mapping $R^n \times R^n \rightarrow R$, which is not defined in a vector space. Among other things, this mapping would provide a mathematical tape measure that could find the distance between two vectors; that is, it could be developed into a norm for calculating the "length" of a vector. The total order relation $<$ could then be defined on the set of elements that make up that vector space.

A very important set of mappings of the form $R^n \times R^n \rightarrow R$ are the metrics, or distances. These functions are measures in the mathematical sense, and associate numbers with such concepts as "length." A *metric* is a real-valued function d defined on $X \times X$ (ordered pairs of elements of a set X). It conforms to the following postulates: $d(A,B) \geqslant 0$ (the distance from one element to another is never negative); $d(A,A) = 0$ (the distance from an element to itself is zero); $d(A,B) = d(B,A)$ (the distance from an element A to an element B is the same as the distance from B to A, hence we speak of the distance between A and B); $d(A,B) > 0$ if $a \neq b$ (the distance between two distinct elements is positive); and $d(A,C) \leqslant d(A,B) + d(B,C)$ (the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides). The metric $d(A,B)$ is a real number, called the *distance* between A and B. This definition of metrics applies to any abstract or numerical set, although in science we deal largely with sets whose elements are numbers or n-tuples of numbers.

We can now define the mathematical systems called *metric spaces* as all sets within which a distance can be computed between any two elements. Anthropology is scarcely the only science in which concepts are defined and evaluated by a "distance between two vectors." As Randall (1968: 41) points out:

> Temperature in chemistry, free energy in quantum mechanics, entropy in thermophysics, selective information in communication theory, and GNP in Western economics all derive from the same general linear metric theory. Concretely, they are measurements of the behavioral output of an entire system. Abstractly, they are the distances between vectors in a metrized linear phase space.

If vectors are to be used as abstractions of empirical data (and no viable alternatives are in sight), then one must partition the general vector spaces from those that are metric spaces as well. This partition can be constructed with the concept of norm; normed vector spaces are also metric spaces. This is so because the norm of the difference of two vectors is a mapping that conforms to the distance postulates: $d(v,w) = \|v - w\|$. This norm, interpreted as a distance, is called the *induced metric on* V. Every normed space associated with the induced metric is a metric space. The mapping $R^n \times R^n \rightarrow R$ is obtained by subtracting from each component of v the corresponding component of w, and taking the norm of the resulting vector as the distance, i.e., the element of R.

Lipschutz (1965: 114) reminds us that metric spaces are not as simple as they may appear to be.

> A metric space is a topological space in which the topology is induced by a metric. Accordingly, all concepts defined for topological spaces are also defined for metric spaces. For example, we can speak about open sets, closed sets, neighborhoods, accumulation points, closure, etc., for metric spaces.

It must be stressed that there exist an unlimited number of metric spaces, defined by exotic distance functions, but that only a very few of these have been interpreted by scientists. Einstein did use an exotic metric with stunning effect to predict the curvature of astronomical space in the neighborhood of a star. But the garden-variety metric invented by Pythagoras and now generalized as the Euclidean metric continues to be more generally useful in science. Like a homely but extraordinarily skillful wife, it is not to be despised. This metric is called the *Euclidean metric on* R^n:

$$d(A, B) = \left[\sum_{i=1}^{n} (a_i - b_i)^2 \right]^{\frac{1}{2}}.$$

The metric space R^n, with the Euclidean metric, is called *Euclidean n-space*, or E^n. Within E^n one can define such familiar geometric entities as distance, angle, area, volume, and even the continuity and convergence to a limit that give us calculus (Abraham 1966: 79).

In computing the Euclidean metric we must multiply two vectors together. Unfortunately, this operation is not defined in any of the mathematical systems we have considered so far. Therefore some form of vector multiplication must be explicitly postulated before Euclidean space can be separated from non-Euclidean spaces. Vector multiplication will be a mapping of the form $R^n \times R^n \to R$. Among the unlimited number of these mappings we select one, called *standard scalar product* and defined by the mapping γ:

$$\gamma : R^n \times R^n \to R : [(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y^n)] \to \sum_{i=1}^{n} x_i y_i .$$

We take corresponding components of two vectors, multiply them together, and then add up all of the products. This will result in a single real number. Thus we have constructed a new mathematical system: $E^n = (R^n, \gamma)$.

It is quite possible to take the same vector, instead of two different ones, and multiply its components by themselves in the same way. It can then be shown that the square root of the sum of these products conforms to the postulates of a norm. In other words, if $x = (x_1, x_2, \ldots, x_n)$, a norm of x can be defined as $\|x\| = [(x_1)^2 + (x_2)^2 + \ldots + (x_n)^2]^{\frac{1}{2}}$. This norm is called the *Euclidean norm on* R^n. It is a generalization of the familiar expression for the distance from the point x

to the origin. In other words, it corresponds to the distance between the zero vector $0 = (0_1, 0_2, \ldots, 0_n)$ and the vector $x = (x_1, x_2, \ldots, x_n)$. We define this distance to be the "length" of the vector x. Measuring the distance between two points of E^n (i.e., implementing the Euclidean metric) involves isolating the line segment connecting both vectors by subtracting one vector's components from the other's; one end of this line segment is then moved down to the origin, and the line's length computed with the Euclidean norm.

Anthropological analyses involve geometric as well as algebraic properties of E^n, to which Gardner (1968) provides an accessible introduction. Consider an n-sphere in E^n, i.e., the locus of points at a given distance from a point:

$$\sum_{i=1}^{n} (x_i)^2 = c\,.$$

A 1-sphere consists of two points on a line on each side of a center; a 2-sphere is a circle in a plane. The surface of an n-sphere has a dimensionality of $(n-1)$; thus a 3-sphere's surface is two-dimensional, whereas a 4-sphere's is three-dimensional. The cross section of a 2-sphere (i.e., a line cutting the circle) is a pair of points; that of a 4-sphere is a 3-sphere. A 4-sphere moving through E^3 would first appear as a point, become a tiny sphere growing to its maximum cross section, and then diminish and disappear. The diagonal of a unit cube in E^2 is the diagonal of a square, and its length is $\sqrt{2}$. Similarly, a line of length $\sqrt{3}$ will fit into a unit cube in E^3, and a ten-foot fishing pole will fit diagonally into a unit cube of E^{100}. Such generalizations cannot be made mechanically, however. The numerical volume of a unit sphere in R^3 is 4.1+; in E^4 it is 4.9+; and in E^5 it is 5.2+. But in E^6 it decreases to 5.1+. In fact, as n approaches infinity, the volume of a unit sphere approaches zero. Anthropological applications of E^n involve a geometric figure called a hyperplane. Consider the points that make up E^2. These are two-component vectors of the form (x_1, x_2). Next, take a pair of real numbers, a_1, a_2. When these are multiplied with the vector components and added together $(a_1x_1 + a_2x_2)$, the sum is another real number, b. There are an unlimited number of vectors in E^2 that conform to the condition

$$\sum_{i=1}^{2} a_i x_i = b$$

and an unlimited number that do not. The vectors that satisfy this function consist of a straight line in E^2. When $n = 3$, the function

$$\sum_{i=1}^{n} a_i x_i = b$$

defines a plane in E^3. When $n > 3$, the function defines a *hyperplane* in E^n. If each vector component measures the number of hours per week spent by a family (or tribe) on a given economic pursuit, and if the corresponding real number measures the number of dollars (or whatever) produced by one hour of that activity, then b measures the total payoff of a week's economic activity.

Once a particular payoff has been observed in the field, one can establish an unlimited number of alternative time allocations that will yield the same payoff as the observed one. The vectors describing these allocations will all lie within (and indeed constitute) one hyperplane:

$$\sum_{i=1}^{3} a_i x_i = b\,.$$

Somewhere in E^n there exists a point (or vector) whose components generate this payoff with the least expenditure of time. This vector represents the maximum integration of the tribe's economy, and can be isolated by linear programming techniques. The "distance" between the observed vector and the vector of maximum economic integration can be computed and used as a measure of cultural integration. In other words, Kroeber's concept of "cultural intensity" can be abstracted as a line segment in E^n (Hoffmann 1966); the "distance" between the beginning and the end of this segment measures the magnitude of cultural intensity. Further anthropological applications of these procedures are being developed with great energy by Buchler (1968).

THE MATRIX $R^{m \times n}$

An empirical event in science often requires more than an n-tuple of numbers for its description. Consider a village social system that involves both age and class distinctions. Its organization at one point in time can be described by listing the number of people in each of the six social states:

	age 10–20	age 21–40	age 41+
social class #1	5	10	6
social class #2	8	40	9

The social system of this village at that point in time can be described mathematically by the rectangular array

$$\begin{pmatrix} 5 & 10 & 6 \\ 8 & 40 & 9 \end{pmatrix}$$

This turns out to be an element of the vector space $R^{2\times 3}$.

We establish this mathematical interpretation of the event by noting that its data can be written on one line: 5, 10, 6, 8, 40, 9. First we run through the three age states of the first social class: x_{11} (=5), x_{12} (=10), x_{13} (=6). Then we list the age states of the second: x_{21}, x_{22}, x_{23}. More generally, one can form an $n \times m$-tuple, $(x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, x_{22}, \ldots, x_{2n}, \ldots, x_{m1}, x_{m2}, \ldots, x_{mn})$, which describes a rectangular array of data made up of n columns and m rows. Such a mathematical expression can be interpreted as an element of an $m \times n$-fold Cartesian product of the real numbers: $R^{m\times n}$. The α mapping can be defined on the set $R^{m\times n}$; empirically, this reflects the fact that data from two villages can be combined by adding the people in corresponding social states. The β mapping can also be defined on $R^{m\times n}$; hence the number of people in each social state can be multiplied by a scalar if the population doubles over a period of time. In other words, the set $R^{m\times n}$ has the algebraic structure of a real vector space: $R^{m\times n} = (R^{m\times n}, \alpha, \beta)$.

An element of this space is called an $n \times m$ *real matrix.* The set-theoretic definition of these matrices is somewhat lengthy (Abraham 1966: 50–59) and will be omitted here. However, we can note two novel properties of the system $R^{m\times n}$. An operation called *matrix multiplication* can be defined by the mapping μ: $R^{m\times n} \times R^{n\times p} \rightarrow R^{m\times p}$. However, the number of columns in matrix A must equal the number of rows in matrix B if the product AB is to be defined. Moreover, matrix multiplication is not necessarily commutative $(AB \neq BA)$. Matrices can be rectangular or square; in the latter case, $m = n$. Square matrices can be mapped into the real numbers by computing their *determinants.*

Events that occur between two points in time are particularly important in science. Consider a tribe whose social structure contains three states (however defined): S_1, S_2, S_3. Next, consider the number of people in each state as the culture is observed at two points in time: S_1^1, S_2^1, S_3^1 and S_1^2, S_2^2, S_3^2. We are interested in such questions as: "How many people in S_1^2 were born of fathers in S_1^1, in S_2^1, and in S_3^1?" Say that these numbers are 10, 55, and 5. This information can be recorded as a column vector of a 3×3 matrix:

$$\begin{array}{c} \\ S_1^1 \\ S_2^1 \\ S_3^1 \end{array} \begin{array}{c} \begin{array}{ccc} S_1^2 & \quad S_2^2 & \quad S_3^2 \end{array} \\ \begin{pmatrix} 10 & \quad & \quad \\ 55 & & \\ 5 & & \end{pmatrix} \end{array}$$

Similarly, the values of the other components can be entered in the array, perhaps giving the matrix

$$\begin{pmatrix} 10 & 25 & 30 \\ 55 & 60 & 35 \\ 5 & 25 & 5 \end{pmatrix}$$

This square matrix is an element of a vector space abstracting one aspect of culture change.

THE MAPPING $R^{m \times n} \times R^{n \times p} \rightarrow R^{m \times p}$

It is entirely appropriate to consider vectors as row matrices, i.e., $R^n = R^{1 \times n}$. These can abstract a variety of anthropological data: the number of people in n social states, the number of man-hours spent on n economic activities, the number of potsherds of each of n types found at one site, etc. If after a period of time, the culture has changed to a different configuration, the new data can again be represented as a space of row matrices. Culture change, then, can also be abstracted as a mapping that takes each element of the first vector space into an element of the second. Such mappings are of the form $R^{1 \times n} \times R^{n \times n} \rightarrow R^{1 \times n}$. Here, $R^{n \times n}$ is interpreted not as an event, but as a *linear transform* that maps one vector space into another. It can be proven that the set of all linear transforms on a vector space is itself a vector space. A good deal of culture theory can probably be abstracted by using this theorem.

The components of $1 \times n$ matrices are sometimes converted to ratios: "10 people in S_1" becomes "10/85 of the 85 people in the tribe belong to S_1." Since the components are defined to exhaust the relevant social states, the ratios will add up to one. Vectors of this kind are called *probability vectors*. The rows of a matrix may be similarly treated, in which case it becomes a *matrix of transition probabilities*. Although the mathematical properties of this subset of $R^{n \times n}$ are more restricted than those of a vector space, they are of extraordinary importance to all the sciences (Bharucha-Reid 1960). A transform of the form

$$\begin{pmatrix} 10/65 & & \\ & \cdots & \\ & & 5/35 \end{pmatrix}$$

will map a $1 \times n$ probability vector into the corresponding vector one time unit in the future. The second power of this transform (the transition matrix multiplied by itself) will map the probability vectors into the corresponding vector two time units in the future. The nth power of this transform, which can be readily computed using a theorem on regular Markov chains, predicts the ultimate fate of any initial probability vector (on the condition that the transition matrix remains fairly stable over time). I have used this structural feature of transition matrices to investigate the stability of Ethiopian age grades (1965), and Buchler has applied it to the Iban *bilek* family (1968: 63–66).

Other subsets of $R^{n \times n}$ are used in anthropology, even though their mathematical structure has not yet been investigated to any great extent. Instead, certain properties of the transforms are established empirically, and it is postulated that these will hold in similar culture-change situations. For example, Dethlefsen and Deetz (1966) have established empirically that the $1 \times n$ matrices of potter-type ratios will indeed produce the battleship-shaped curves postulated in seriation (1966). Elsewhere, these authors also point out that the curves will appear somewhat different to archaeologists moving in different directions or at different time rates from one another (Deetz and Dethlefsen 1965). In any event, the transforms that map one site into the next in time can reduce the difference between components of the corresponding vectors of type ratios to a minimum. From this empirically validated postulate the Aschers have developed a computer program that orders a collection of sites chronologically (1963: 1045–52).

Computer simulations of an entire social system may be interpreted as the operation of $R^{n \times n}$ transforms on an extensive matrix of social categories. The transforms are assembled from a variety of heterogeneous components, such as life expectancies, population data, and mathematical functions that abstract marriage rules. These are built into a transform by the aptly named Monte Carlo method. Unfortunately, the mathematical structure of these transforms is very complex, and they cannot easily be applied to more than one problem (Gilbert and Hammel 1966).

A NOTE ON THE MATHEMATICS OF KIN-TERM FUNCTIONS

In recent months, several advances have been made in the semantic analysis of kinship lexicons, i.e., of kin-term functions.* These are mappings of kin types onto a set of kin terms, mappings of lexical items onto semantic points (Kay and Romney 1967: 13; *see also* Wallace and Atkins 1960: 70, and Kay 1966: 20), or other mappings on a set of kin terms. Kay (1968: 221–22, 253–54) has developed a function that generalizes the distinctions between Dravidian cross and parallel. Sanday (1968) has used information-processing theory to model kin-term naming behavior.

Since mathematical theory has evolved primarily as a Euro-American conceptual tool, it would be foolish to assume that Iroquois, Omaha, and other kin-term functions can necessarily be structured by any known mathematics. Conventional mathematics may be useless in constructing isomorphic or homomorphic models of kin terminology; if so, the several existing foundations for an anthropology-specific mathematics may be more helpful (e.g., Romney and D'Andrade 1964, Lamb 1965: 56, Hammel 1965). On the other hand, kin-term functions may be unrecognized varieties of previously studied mathematical functions (or mappings). In this case, a formal identi-

* This section was contributed by Robert A. Randall, University of California at Berkeley.

fication of the appropriate class of mathematical mappings would be an invaluable aid to further analysis.

In mathematics, it is usual to characterize a space by the mappings defined on a set. A semantic space for kin terminology would be a set of kin terms on which various functions are defined. This was evidently Wallace and Atkins's meaning (1960: 70) when they defined a semantic space as a group of logical predicates related by certain logical rules, "mapping of particular sets of terms on semantic space." A paradigmatic mapping involves three "Boolean" operations: set union $\cup$ (and/or), set intersection $\cap$ (and/also), and set complementation $\sim$ (not). These set operations are fundamental to mathematical logic, and they appear to be almost as useful in semantic analysis (cf. Kay and Romney 1967). At first glance, the structure of semantic spaces might appear well-defined, especially if one accepts Wallace and Atkins's references to "algebra" and "Boolean algebra" (1960: 62) uncritically. But in fact, Wallace and Atkins employ the weakest of all structuring tools, the algebra of sets. The very limited payoffs that derive from this choice will become more evident if we review the several mathematical meanings of the term "algebra."

In its most general sense, an algebra is any non-empty set, together with a binary operation (o) from $A \times A$ into A (Suppes 1957: 252). Wallace and Atkins gave kin terminology an algebraic structure because "the algebra of sets" includes two binary operations ($\cup$ and $\cap$) and the "unary" operation of complementation. However, unrestricted set operations are so general that most mathematicians do not accord them structuring capabilities; instead set operations are simply assumed in all meaningful mathematical discourse. For this reason, set theory is an insufficient structuring tool for semantic space. But Wallace and Atkins do not suggest any other structure. In particular, their semantic space is not actually structured by Boolean algebra.

A Boolean algebra of sets is a non-empty class A of sets (in a universe U) that is closed under the set operations of union, intersection, and complementation. With these restrictions, if "male and also lineal and also first ascending generation" is a kin term, then "not male and also not lineal and also not first ascending generation" must be a kinship term as well. Since this and other closure transformations by Boolean operators are not found in familiar kinship lexicons, Boolean algebra is an inappropriate structure. Hammel (1965: 65) realized this when he suggested that the term "kinship algebra" is inexact because semantic analyses usually deal with "description, not manipulation." Semantic spaces should be structured by the logic systems that they model, not by the kin-term sets themselves. Although sets of terms have many descriptions under the algebra of sets, they have far fewer when both terms and operations between terms are modeled. Wallace and Atkins (1960: 74) did suggest a further possibility: paradigms with the non-Boolean operator of genealogical relation ("of": $Si\ o\ Fa \rightarrow FaSi$) will have a different structure from those composed solely of conjunctive and disjunctive operators. However, the authors did not develop this idea.

Wallace and Atkins have further confused their space structure by an unfortunate analogy with the third sense of the term "algebra." Algebraic spaces are subtypes of the class of spaces (L) called linear (vector) spaces (Simmons 1963: 208).* It happens that the mathematical theory of dimensions makes mathematical sense only when the structure involved is linear. Briefly, let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite, non-empty set of vectors. If there do not exist non-zero numbers (α_i) such that $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$, then S is said to be linearly independent. The largest such independent subset S_{max} in L is called the basis of L (*ibid.*: 196). The number of elements in S_{max} is the dimension of L (*ibid.*: 200). This concept of dimension corresponds closely to the notion normally employed in Euclidean geometry.

Since algebraic spaces are also linear spaces, they possess mathematical dimensions. However, Wallace and Atkins (1960: 70) do not follow this usage. Rather, features ($d_i, d_j, \ldots, d_n$) in mutually exclusive subsets (α_i) of a set of empirical phenomena (A), are said to be part of a single dimension (D), if they are contraries to one another. By this criterion, at least one dimension on A must be logically independent of at least one another, but the remaining dimensions can be logically dependent. Semantic dimensions are not required to be linearly independent, nor is the space structure really asserted

* See also the discussion on pp. 58–62.

to be algebraic or even linear. In short, the authors do not establish a general relation between semantic dimensions and mathematical dimensions, nor even a very close analogy.

In mathematical terminology, Wallace and Atkins's semantic spaces are not mathematical spaces, but form what is usually called a product set. Each semantic dimension is actually a coordinate set, and the individual values on these dimensions (i.e., features) are coordinates (cf. Simmons 1963: 24). Mathematically, a paradigm is a mapping of a set of terms into a semantic product set ($X = X_1 \times X_2 \times \ldots \times X_n$), not into a semantic space. Each element of this semantic set is not a vector, but an n-tuple of coordinates $x = (x_1, x_2, \ldots, x_n)$ somewhat like Kay and Romney's semantic point (1967: 6); hence the individual coordinates (x_i) are not vector components, nor are their containers (the X_i's). Semantic components and vector components have quite different referents: the first refer to features (coordinates) on a semantic dimension (a coordinate set); the second refer to coordinates on a linear dimension (a coordinate set of a linear space).

These distinctions would be trivial terminological quibbles if Aoki (1966) and others (Buchler and Selby 1968: 171ff) had not used "matrices" (column n-tuples) as a means of performing Lounsbury-like "transformational analyses" on Nez Percé, Omaha, and Iroquois terminological systems. These authors suggest that transformation rules can be viewed as binary operations on bundles of features (i.e., pairs of n-tuples). Since these operations produce an algebraic structure, they necessarily produce a space. In investigating kin-term functions, we can try to probe the structure of this space; in particular, we can try to discover whether the Aoki-Buchler-Selby separation of kin types into features corresponds to the algebraic structure generated by their selected binary operation.

Mathematical separation properties are studied in topology. Hence we can use topology to discover whether various separations of kinsmen by semantic features (Aoki 1966: 361, Buchler and Selby 1968: 171) correspond to separations in the structure of the space. We must first define a topology on a set of kin terms structured by genealogical relations, and then compare the properties of the topology to the properties of the semantic analysis. Briefly, a topology (T) is a class of subsets of a non-empty set X for which (1) the arbitrary union of every class of sets in T is a set in T and (2) the intersection of every finite class of sets in T is a set in T (Simmons 1963: 92). A set X containing a topology is called a topological space, and its subsets are "open sets." A topological space that cannot be represented as a disjoint union of two non-empty open sets is said to be *connected* (*ibid.*: 143). This definition of connectedness corresponds closely to the intuitive concept of "a piece" in Euclidean geometry. A subspace of a topological space is called a *topological component* if and only if it is connected and is contained in no larger connected topological subspace (*ibid.*: 146). In short, there is a qualitative difference between the structure of topological components and that of surrounding space.

Boyd (1965: 5) recognized that semantic components derive their reality from the space structure in which they exist. In investigating Kariera and Ambryn marriage classes and American kin terminological systems, he used a partitioning technique called the substitution property to show that some components of these systems are dependent on others. This ordering relation redefines the variables in the semantic space (the components) as entities that possess a high degree of structural coherence under the operations of the space. With Boyd's ordering relation, anthropologists can build a semantic space from well-known mathematical functions, since ordering properties alone are enough to convert semantic components into topological components. For example, assume that the consanguineal Fox semantic components of sex (S), lineage (L), and generation (G) are placed under the binary operation of genealogical relation. It can be shown that a Fox genealogical rule for predicting kin terms is fully determined (i.e., single-valued) only if the generation semantic feature depends for its value on the lineage feature, and the lineage feature depends for its value on the sex feature ($S \rightarrow L \rightarrow G$). Since operations are fully determined (by definition), the structure of the semantic space necessitates the modeling of this dependency.

These constraints become more obvious if we consider two alternative interpretations of this same Fox structure. Suppose certain features of the ordering relation are selected to produce the homomorphic model of the Fox system that is called a complemented distributive lattice. This structure can be shown to be a Boolean algebra of sets (Simmons 1963: 345) that is isomorphic to a subtype of a space with the separation properties of the "totally disconnected" Hausdorff topology (*ibid.*: 353). Under this topology, each semantic component of Fox (S, L, or G) is a topological component, but the dependency between semantic components is not preserved because neither Boolean algebra nor lattice theory abstracts it. By contrast, the most interesting aspect of Fox and other "Omaha" systems (generational skewing) is preserved by the so-called "ordering topology" (cf. Kelley 1955: 58). In this space, the topology $T = \{(S,L,G), (S,L), (S), \emptyset\}$ is connected, since it cannot be disjointly partitioned; thus the entire semantic space has only one topological component. In short, isomorphic models of particular systems must be homomorphically modeled by the ordering topology. Without this structure, manipulations of feature bundles will not be unique, and hence will not be transformations. With this structure, kin-term functions will generate a mathematically regular semantic space.

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