Applied Math Modeling Midterm Exam (Spring 2018)

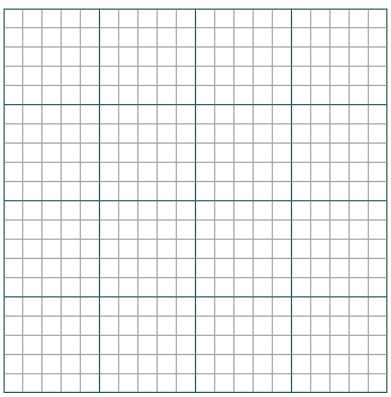
Name:

Directions: Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). You **must skip** one problem (other than the first). Write "skip" clearly on the one you skip. **Good luck!**

Problem 1. (10 pts) (You can't skip this one!) Use the following data

x	-2	-1	0	1	2
y	-20	10	0	-10	20

a. (8 pts) to construct the simple linear regression model of best fit (find $y(x) = a + bx$). Show all work. You might sketch the data and your line in the grid provided.



b. (2 pts) Also compute R^2. Show all work.

Problem 2. (10 pts) We like linear functions, operations, etc. in mathematics, because they tend to make our lives easy. Illustrate **graphically** how Newton's method is an example of using a linear method to solve a non-linear problem in the case of finding the root of $\ln(x)$ from a starting point of $x_0 = \frac{1}{2}$. I suggest a "plot window" of $[0, 2] \times [-2, 2]$, using the entire grid below.

No need to go beyond the first step of Newton's method. Describe as explicitly as you can

a. (3 pts) the non-linear problem,

b. (4 pts) the associated linear problem, and

c. (3 pts) describe how we make use of the linear problem to solve the non-linear problem.

Problem 3. (10 pts) I have issues with some regressions: I hope that you do, too!

a. Comment on the linear regression model featured in this graphic, and sketch in a model you might prefer (justify).

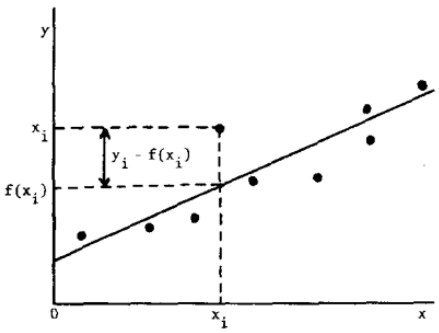
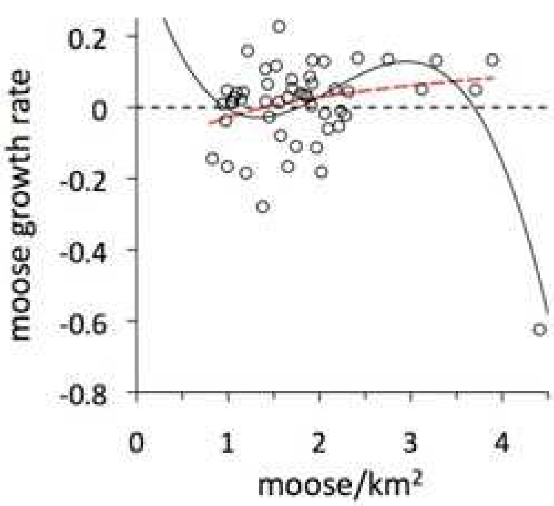


Figure 5. Deviations

b. Comment on the linear regression model featured in this graphic (from "study.com" – and it's not the squiggliness of the "line" that bothers me, or that it was the second hit googling regression images). Sketch in a model you might prefer (justify).

c. Reflect a little on the linear regression model(s) featured in this graphic: the dashed curve is one regression, whereas the solid curve (a cubic) is called "the most parsimonious relationship between moose abundance and population growth rate". What concerns might you have?



Problem 4. (10 pts)

a. For the Keeling data, with data given monthly (but assume time t measured in years),

i. (2 pts) what precise pair of trigonometric functions did we use (along with a quadratic trend model) to model **the oscillations** in the data?

ii. (2 pts) Why did we use **a pair** of such functions?

iii. (2 pts) Given the pair $a\sin(x)$ and $b\cos(x)$, what is the **amplitude** of the combined oscillation?

b. For the sea anemone data, Duration of Response time as a function of Anemone size,

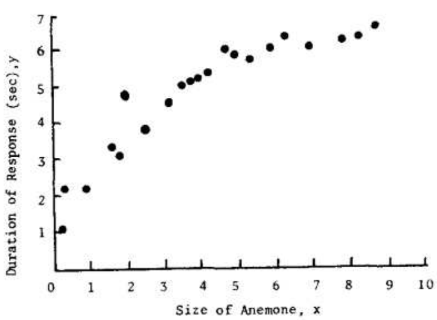


Figure 2. Response of Sea Anemones to Standard Stimulus

FittedModel[$2.94754\,x^{0.390763}$]

	Estimate	Standard Error	Confidence Interval
a	2.94754	0.15844	{2.61592, 3.27916}
b	0.390763	0.0324306	{0.322885, 0.458641}

i. (2 pts) what **structural** idea – that is, characteristic of anemones – made us propose a model of the form $Duration(x) = a\sqrt{x}$ (based on the unspecified "size" of the anemone)?

ii. (2 pts) Power model regression produced the results shown (95% confidence). What do the results suggest about our proposed model?

Problem 5. (10 pts) One change we made to the equations for the Moose/Wolf compartmental model was to replace one Lotka-Volterra birth process by a logistic growth process: that is,

$$\begin{aligned}\frac{dm}{dt} &= b_m m(t) - d_m m(t)w(t) \\ \frac{dw}{dt} &= b_w m(t)w(t) - d_w w(t)\end{aligned} \quad \longrightarrow \quad \begin{aligned}\frac{dm}{dt} &= b_m m(t)\left(1 - \frac{m(t)}{K}\right) - d_m m(t)w(t) \\ \frac{dw}{dt} &= b_w m(t)w(t) - d_w w(t)\end{aligned}$$

where b_m, d_m, b_w, and d_w are constants.

a. (1 pts) What do we call K?

b. (2 pts) What justification did we have for making that change?

c. (2 pts) Describe the dynamics of the moose population in the absence of wolves. Consider both cases: initial population values below or above K.

d. (5 pts) Find the **non-zero** populations of wolves and moose that are **equilibria** for this new system. (Hint: start by considering the $\frac{dw}{dt}$ equation, which didn't change.)

Problem 6. (10 pts)

a. (6 pts) Demonstrate that one can estimate the coefficients in the model $y(t) = ax^b$ by linearization: that is, that one can estimate the coefficients a and b of the non-linear model from data $\{x_i, y_i\}_{i=1,\ldots,n}$ using **linear** regression on transformed data, then back-transforming the parameters from the appropriate linear model, α and β.

   Identify how you would transform the data for use in the linear regression.

b. i. (2 pts) How do you back-transform to obtain estimates for a and b?

   ii. (2 pts) What important change occurs in the confidence intervals obtained from linear regression if you back-transform them in the same way you back-transform the parameters? How do they differ from traditional CIs for linear regression?

Problem 7. (10 pts) In class we demonstrated that the simple linear regression line must pass through the center of mass of the data, $(\overline{x}, \overline{y})$. Thus there is really only one "degree of freedom" for the line – the slope m. Use a little univariate calculus to **derive** the value of m, by minimizing

$$f(m) = \sum_{i=1}^{N} (y_i - \overline{y} - m(x_i - \overline{x}))^2$$