MAT 229: An Infinite Series Toolbox

(rev. March 2019 – Thanks to Roger Zarnowski!)

Following is a summary of results for infinite series, from chapter 11 of the text. Material from section 11.1 on sequences is also very important, although not included here.

General Tools

- **Analogy with Improper Integrals**

Improper Integrals (Type I)	Infinite Series
If f is continuous with antiderivative F, $$\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t f(x)\,dx = \lim_{t\to\infty}\left(F(x)\big\|_1^t\right),$$ provided the limit exists.	Define $S_N = \sum_{k=1}^N a_k = a_1 + a_2 + \cdots + a_N$. Then $$\sum_{k=1}^\infty a_k = \lim_{k\to\infty}\sum_{k=1}^N a_k = \lim_{N\to\infty} S_N (\text{a limit of a } \textit{sequence})$$ provided the limit exists.

- **Definition:**

$$\sum_{k=1}^\infty a_k = \lim_{N\to\infty}\sum_{k=1}^N a_k$$

provided this limit exists as some real number S. In that case the infinite series is said to be *convergent*, with value S. If the limit on the right does not exist, the infinite series is *divergent*.

The expression $\sum_{k=1}^N a_k$ is denoted S_N and is called the N^{th} **partial sum**. Convergence of the infinite *series* $\sum_{k=1}^\infty a_k$ is therefore defined as convergence of the *sequence* of partial sums $\{S_N\}$.

NOTE: Unfortunately, S_N usually can't be evaluated in closed form, and so the above limit often cannot be determined analytically. See the next section for two special cases in which this *can* be done.

- **The Divergence Test** (sometimes called **the nth-term test**)

If $\lim_{k\to\infty} a_k \neq 0$ then $\sum_{k=1}^\infty a_k$ is divergent. (You should know how to prove this, although the proof uses the equivalent contrapositive statement, which says that if $\sum_{k=1}^\infty a_k$ is convergent, then $\lim_{k\to\infty} a_k = 0$.)

NOTE: If $\lim_{k\to\infty} a_k = 0$ then the series $\sum_{k=1}^\infty a_k$ may either be convergent (as with a convergent p-series) or divergent (as with the harmonic series). Be sure you understand the logic of these statements.

Series for which S_N can be computed in closed form

- *Geometric series:*

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots ,$$

where $a \neq 0$. Then

$$\begin{aligned} S_N = \sum_{k=0}^{N} ar^k &= a + ar + ar^2 + \cdots + ar^N \\ &= \begin{cases} Na, \text{ if } \ r = 1 \\ a\frac{1-r^{N+1}}{1-r}, \text{ if } \ r \neq 1. \end{cases} \end{aligned}$$

If $|r| \geq 1$ the series diverges since $\lim_{N\to\infty} S_N$ does not exist.

If $|r| < 1$ then $\lim_{N\to\infty} S_N = a\dfrac{1-0}{1-r} = \dfrac{a}{1-r}$ and the series therefore converges to $\dfrac{a}{1-r}$.

NOTE: Geometric series may appear in forms other than the general one shown above, but they can always be converted to that form. In any case, if $|r| < 1$ the series converges to $\frac{\text{first term}}{1-r}$. (See examples from class or in the text.)

- *Telescoping series:*

$$\sum_{k=1}^{\infty} (c_k - c_{k+1}) = (c_1 - c_2) + (c_2 - c_3) + \cdots$$

In this case,

$$\begin{aligned} S_N &= \sum_{k=1}^{N} (c_k - c_{k+1}) \\ &= (c_1 - \cancel{c_2}) + (\cancel{c_2} - \cancel{c_3}) + \cdots + (\cancel{c_{N-1}} - \cancel{c_N}) + (\cancel{c_N} - c_{N+1}) \\ &= c_1 - c_{N+1} \end{aligned}$$

NOTE: Telescoping series may not initially appear in the form shown, but must usually be rewritten into that form, for example, by using partial fraction decompositions, properties of logarithms, etc. (See examples from class or in the text.)

Tools for Series of Positive Terms

- **The Integral Test**

If $f(x)$ is continuous, positive-valued, and decreasing on $[1, \infty)$, and if $a_k = f(k)$ for $k = 1, 2, 3, \ldots$, then the infinite series $\displaystyle\sum_{k=1}^{\infty} a_k$ and the improper integral $\displaystyle\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

- **p-series**

By applying the Integral Test to the function $f(x) = \dfrac{1}{x^p}$, we find that the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

- **Error Bounds for Approximating *Series of Positive Terms* by Partial Sums**

  Suppose f and a_k are as described in the hypotheses of the integral test and suppose we know that $\sum_{k=1}^{\infty} a_k$ is convergent (either by the integral test or some other test). Let S denote the exact (unknown) value of the sum of the series, and suppose we approximate S by the partial sum S_N. The error in this approximation is R_N where $S = S_N + R_N$, or $R_N = S - S_N$, as shown below:

$$\begin{aligned} S &= \sum_{k=1}^{\infty} a_k \\ &= \underbrace{a_1 + a_2 + \cdots + a_N}\; + \;\underbrace{a_{N+1} + \cdots} \\ &= S_N(\text{approximation}) + R_N(\text{error}). \end{aligned}$$

  Then

$$\int_{N+1}^{\infty} f(x)\,dx \le R_N \le \int_{N}^{\infty} f(x)\,dx.$$

  This gives bounds on the size of the error R_N. Since $R_N = S - S_N$, this can be rewritten to give bounds on the exact value of the series, as follows (the "100% confidence box"):

$$S_N + \int_{N+1}^{\infty} f(x)\,dx \le S \le S_N + \int_{N}^{\infty} f(x)\,dx.$$

- **The Direct Comparison Test**

  1. If $0 < a_k \le b_k$ for all k and $\sum_{k=1}^{\infty} b_k$ is convergent, then so is $\sum_{k=1}^{\infty} a_k$.

     (If the "larger" series is convergent, then so is the "smaller" one.)

     In this case, the test would be applied by finding a *known convergent* series with terms b_k in order to prove convergence of the series that has terms a_k.

  2. If $0 < a_k \le b_k$ for all k and $\sum_{k=1}^{\infty} a_k$ is divergent, then so is $\sum_{k=1}^{\infty} b_k$.

     (If the "smaller" series is divergent, then so is the "larger" one.)

     In this case, the test would be applied by finding a *known divergent* series with terms a_k in order to prove divergence of the series that has terms b_k.

- **The Limit Comparison Test**

  If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series of positive terms and if $\lim_{k\to\infty} \frac{a_k}{b_k}$ is a positive real number (not 0 and not ∞) then either both series converge or both series diverge.

  We may sometimes be able to draw a conclusion even if the the limit is 0 or ∞ :

  - If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum_{k=1}^{\infty} b_k$ is convergent, then so is $\sum_{k=1}^{\infty} a_k$.

- ∘ If $\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$ and $\sum_{k=1}^{\infty} b_k$ is divergent, then so is $\sum_{k=1}^{\infty} a_k$.

Tools for Alternating Series

- **The Alternating Series Test (AST)**

  Consider $\sum_{k=1}^{\infty}(-1)^{k+1}b_k = b_1 - b_2 + b_3 - b_4 + \ldots$, where $b_k > 0$ for all n. If the following two conditions hold:

  (i) $\lim_{k\to\infty} b_k = 0$

  (ii) $b_{k+1} \leq b_k$ for all k greater than some integer M, (i.e., the terms are **eventually** decreasing – remember, it's all about the tails)

  then the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}b_k$ is convergent. (The same holds for $\sum_{k=1}^{\infty}(-1)^n b_k$.)

  NOTE:

  1. Verification of the inequality in (ii) is usually accomplished in one of two ways: either by algebraic simplification (cross-multiplying, etc.), or by identifying a function f for which $b_k = f(k)$ and showing that $f'(x) < 0$.

  2. If the alternating series can be shown convergent by the AST, it still leaves open the question of whether the series is absolutely convergent (AC) or conditionally convergent (CC). "Absolute convergence" means that the series of absolute values converges – this is the strongest type of convergence, such as with alternating p-series with $p > 1$. "Conditional convergence" means that the alternating series converges only because of the cancellations between positive and negative terms, such as with the alternating harmonic series.

- **Error Bounds for Approximating *Alternating Series* by Partial Sums**

  Consider the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}b_k$. Suppose that $\lim_{k\to\infty} b_k = 0$ and $b_{k+1} \leq b_k$ for all k. Then, by the AST, the series is convergent. Let S denote the exact (unknown) value of the sum of the series, and suppose we approximate S by the partial sum S_N. The error in this approximation is R_N where $S = S_N + R_N$, or $R_N = S - S_N$, as shown below:

$$\begin{aligned} S &= \sum_{k=1}^{\infty}(-1)^{k+1}b_k \\ &= \underbrace{b_1 - b_2 + \cdots + (-1)^{N+1}b_N}_{} + \underbrace{(-1)^N b_{N+1} + \cdots}_{} \\ &= \quad S_N(\text{approximation}) \quad + \quad R_N(\text{error}). \end{aligned}$$

  Then

$$|R_N| \leq |b_{N+1}|,$$

  that is, the error is bounded by the size of the first neglected term.

Absolute Convergence Test

If $\sum_{k=1}^{\infty} |a_k|$ converges, then so does $\sum_{k=1}^{\infty} a_k$.

(and we say that the series is **absolutely convergent**). If a series is absolutely convergent, then the terms of the series may be re-arranged in any order, and they will always give the same sum.

The Ratio and Root Tests

These tests apply to *any* infinite series:

- **The Ratio Test**

  (i) If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$ then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent (AC).

  (ii)) If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$ or if this limit is ∞, then the series $\sum_{k=1}^{\infty} a_k$ is divergent.

  (iii) If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 1$ then this test is inconclusive.

- **The Root Test**

  (i) If $\lim_{k\to\infty} \sqrt[k]{|a_k|} < 1$ then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent (AC).

  (ii)) If $\lim_{k\to\infty} \sqrt[k]{|a_k|} > 1$ or if this limit is ∞, then the series $\sum_{k=1}^{\infty} a_k$ is divergent.

  (iii) If $\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1$ then this test is inconclusive.