Section Summary: The Integral Test

1 Theorems

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series

$$S = \sum_{n=1}^{\infty} a_n$$

is convergent if and only if the improper integral

$$I = \int_1^{\infty} f(x)dx$$

is convergent. If I diverges, then S diverges (and *vice versa*).

Note that it is not essential for f to be positive and decreasing everywhere, but it must be *ultimately positive and decreasing* (that is, decreasing beyond some fixed value of x).

The **p-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent if $p > 1$, and divergent if $p \leq 1$.

If $\sum a_n$ converges by the integral test, with sum S, and the **remainder** $R_n = S - S_n$ (with S_n the

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

(this gives us a bound on the error we're making in the calculation of a series). Note that

$$R_n = \sum_{n+1}^{\infty} a_n$$

2 Summary

Integrals serve as useful tools for evaluating series, determining whether series converge, and even in estimating the error we're making in an estimate of the limit of a series.

Notice that the integral test itself just tells us that a series either converges or diverges – it doesn't give us the value itself. But that, for those series that converge, we can use the integral to bound the error we make in using the n^{th} partial sum S_n to **approximate** the series.