Section Summary: the cross product

a. Definitions

We create the **cross-product** to be a sort of "right-hand rule" operator. We might start by saying that we want the cross-product to have the following results:

$$\hat{\imath} \otimes \hat{\jmath} = \hat{k}$$
$$\hat{\jmath} \otimes \hat{\imath} = -\hat{k}$$
$$\hat{\jmath} \otimes \hat{k} = \hat{\imath}$$
$$\hat{k} \otimes \hat{\jmath} = -\hat{\imath}$$
$$\hat{\imath} \otimes \hat{k} = -\hat{\jmath}$$
$$\hat{k} \otimes \hat{\imath} = \hat{\jmath}$$

Now assume the usual sorts of distributive properties, so that given $\mathbf{a}= \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b}= \langle b_1, b_2, b_3 \rangle$, we can compute

$$\mathbf{a} \otimes \mathbf{b} = \langle a_1, a_2, a_3 \rangle \otimes \langle b_1, b_2, b_3 \rangle = (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \otimes (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$$

or

$$\mathbf{a} \otimes \mathbf{b} = (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_1b_3)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}).$$

Note that the cross-product is a vector-product, by contrast with the dot product which produces a number (scalar). Furthermore, the direction of the cross-product is determined by using the right-hand rule.

This cross-product is only defined for three-dimensional vectors, by contrast with the dot product, which is defined for all vectors in all dimensions.

b. Theorems

$$\mathbf{a} \otimes \mathbf{a} = \mathbf{0}$$
$$\mathbf{a} \otimes \mathbf{b} \perp \mathbf{a}$$
$$\mathbf{a} \otimes \mathbf{b} \perp \mathbf{b}$$
$$\mathbf{a} \otimes \mathbf{b} = -\mathbf{b} \otimes \mathbf{a}$$
$$|\mathbf{a} \otimes \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. This last property can be interpreted geometrically: the norm (length) of the cross product $\mathbf{a} \otimes \mathbf{b}$ is the area of the parallelogram constructed using vectors $\mathbf{a}$ and $\mathbf{b}$.

This can be generalized to the parallelpiped, constructed of three vectors, whose volume V is given by

$$V = |\mathbf{c} \cdot (\mathbf{a} \otimes \mathbf{b})|$$

If we discover that V is zero, then the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ must be coplanar (the parallelpiped is at most two-dimensional). Note: by symmetry,

$$V = |\mathbf{c} \cdot (\mathbf{a} \otimes \mathbf{b})| = |\mathbf{a} \cdot (\mathbf{b} \otimes \mathbf{c})| = |\mathbf{b} \cdot (\mathbf{c} \otimes \mathbf{a})|$$

Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \otimes \mathbf{b} = \mathbf{0}$$

c. Properties/Tricks/Hints/Etc.

d. Summary

Thus the cross-product of two vectors produces a vector which is perpendicular to those two vectors, having magnitude the area of the parallelogram created by the two vectors. We can use this product as a means for determining when two vectors are parallel (they have a zero – zero **vector** – cross-product).