Section Summaries: Power/Taylor Series

1 Definitions

power series: A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

where the c_i are called the **coefficients** of the series. More generally,

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

is called a **power series in** $x - a$, or a **power series centered at** a, or a **power series about** a.

Consider a power series about a: the **radius of convergence** is the largest number R such that the power series converges for all x in

$$|x - a| < R$$

The endpoints often need to be tested separately, and the resultant **interval of convergence** is the interval for which the power series is defined (possibly including one or both of the endpoints).

Taylor series of f about a (assuming f has derivatives of all orders):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The n^{th}**-degree Taylor polynomial of f at** a:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Then $R_n(x) = f(x) - T_n(x)$ is the called the **remainder** of the Taylor series.

Maclaurin series: a Taylor series centered about $x = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

2 Theorems

For a given power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

there are only three possibilities:

a. The series converges only when $x = a$ (radius of convergence $R = 0$);

b. the series converges only when $|x - a| < R$ (radius of convergence R), and possibly at the endpoints, and diverges otherwise; or

c. the series converges for all real numbers x (radius of convergence $R = \infty$).

term by term differentiation and integration If the power series

$$\sum c_n (x-a)^n$$

has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=1}^{\infty} c_n (x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

a.

$$f'(x) = c_1 + 2c_2(x-a) + \ldots = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$$

b.

$$\int f(x)dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + \ldots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the integrated and differentiated power series are both R.

We do have to check the endpoints of the interval of convergence however: differentiation tends to roughen things up, so if an endpoint is convergent for f, it may not be for f'; whereas integration tends to smooth things out, so if an endpoint is divergent for f, it may be "healed" by the anti-derivative of f.

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

If $f(x) = T_n(x) + R_n(x)$, and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Taylor's inequality: If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for} \quad |x-a| \le d$$

3 Properties, Hints, etc.

"In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test." (p. 768)

Power series can be added and subtracted just like polynomials (but be aware of possibly different intervals of convergence). While they can also be multiplied and divided like polynomials, they're quite cumbersome to manipulate this way. We're often only interested in the first few terms, however, which makes this an occasionally useful option.

Some important Maclaurin series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (-1,1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty,\infty)$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad (-\infty,\infty)$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad (-\infty,\infty)$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad [-1,1]$$

4 Summary

We can think of power series as infinite polynomials: very oddly defined functions! Of course we'll wonder about the domain of these functions, which is fundamentally tied to the issue of convergence. We'll say that these functions are defined for any value of x for which the series converges.

We'll use these functions to provide useful approximations to functions (by only taking a partial sum), meaning that it will be possible to approximate functions by **finite** – and hence ordinary – polynomials.

If we want to approximate a function in the vicinity of a certain point a, then we will center our power series about a.

We can construct new power series from old ones in several ways: by

a. writing them as composite functions,

b. integration

c. differentiation

The radius of convergence in the last two cases stays the same as the original power series; in the case of composition, we need to recalculate the radius of convergence.

The interval of convergence does not necessarily remain the same, however: in particular, you still have to check the ends separately, as noted above.

Notice how power series can be used to integrate (approximately) complicated function, and then provide a mechanism for determining the error in the approximation.

Taylor series is the crowning glory of sequences and series. This material clarifies statements like these: "$\sin(x) \approx x$ about $x = 0$." Or, equivalently,

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

We can obtain Taylor series by term-by-term differentiation: suppose that

$$f(x)=\sum_{n=0}^{\infty}c_nx^n$$

with radius of convergence $R>0$. Then $f(0)=c_0$ (the series is easy to evaluate at $x=0$!). We've seen that we can differentiate series, so, provided f is differentiable,

$$f^{'}(x)=\sum_{n=1}^{\infty}c_nnx^{n-1}$$

Again, evaluate at 0: $f^{'}(0)=c_1\cdot 1=c_1$. So $c_1=f^{'}(0)$.

Continuing,

$$f^{''}(x)=\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}$$

Again, evaluate at 0: $f''(0)=c_2\cdot 2\cdot 1=2c_2$. So $c_2=\frac{f''(0)}{2}$.

In general,

$$c_n=\frac{f^{(n)}(0)}{n!}$$

and

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

This is the so-called "Maclaurin series" (a special type of Taylor series, expanded about $x=0$).

We can also obtain Taylor series by building up: the objective is to approximate a function about an abscissa a as well as possible by successively larger polynomials. If you only had a constant function to work with, you'd choose $p(x)=c=f(a)$. If you had a linear function to work with, you'd choose $p(x)=ax+b=f^{'}(a)(x-a)+f(a)$ (check that this gets both the function value right, and the slope right). Continuing in this fashion, you'd get

$$\frac{f^{''}(a)}{2}(x-a)^2+f^{'}(a)(x-a)+f(a)$$

$$\frac{f^{'''}(a)}{3!}(x-a)^3+\frac{f^{''}(a)}{2}(x-a)^2+f^{'}(a)(x-a)+f(a)$$

etc.