Taylor Polynomials

Taylor series review

If function $f(x)$ has a power series representation centered at a, then that power series must be the Taylor series centered at a,

$$\text{f(x)}=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Question

Let $f(x)=\sin(x)$.

- What is the Taylor series for $f(x)$ centered at π?
- What is its interval of convergence?
- We have already seen that the Taylor series for $f(x)$ centered at 0 is $\sin(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$. Replace x in this with $x-\pi$. How do the results of this series compare with the Taylor series for $f(x)$ centered at π?
- There is a trigonometric identity that $\sin(x-\pi)=-\sin(x)$. What does this mean for the above calculations?

Tangent line approximation

Question

Given a general function $f(x)$ and value $x=a$, what is the tangent line approximation of $f(x)$ at a?

Questions

Let $f(x)=\cos(x)+\sin(2x)$.

- What is an equation of the tangent line to $y=f(x)$ at $x=0$?
- What is the sum of the linear terms of the series for $\cos(x)$ and $\sin(2x)$ at 0?

Taylor polynomials

Definition

The n^{th} **degree Taylor polynomial of** $f(x)$ **centered at** a is the partial sum of the Taylor series that goes up to and includes the n^{th} power of $(x - a)$. If the Taylor series for $f(x)$ centered at a converges to $f(x)$ for a given value x, then the n^{th} Taylor polynomial of $f(x)$ centered at a provides a polynomial approximation to $f(x)$.

Questions

Let $f(x) = \cos(x) + \sin(2x)$.

- What is the first degree Taylor polynomial of $f(x)$ centered at 0?
- What is the second degree Taylor polynomial of $f(x)$ centered at 0?
- What is the third degree Taylor polynomial of $f(x)$ centered at 0?
- Plot the graphs of three polynomials along with the graph $y = f(x)$.
- Using the third degree Taylor polynomial of $f(x)$ centered at 0, approximate $f(0.5)$.

Taylor remainder

If you approximate a quantity, you need some way to analyze how good the approximation is. Consider the error = | approximation − exact |. Here the exact is a given function $g(x)$ and the approximation is the n^{th} degree Taylor polynomial of $g(x)$ centered at a.

Definition

The n^{th} **Taylor remainder of** $g(x)$ **centered at** a is

$$R_n(x) = g(x) - \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!} (x - a)^k$$

In other words the error in using the n^{th} degree Taylor polynomial to approximate the function is error = $|R_n(x)|$

Analyzing error

The Taylor series error estimate: If $\left| f^{(n+1)}(x) \right| \leq M$ for all values of x of interest, then

$$\left| R_n(x) \right| \leq \frac{M}{(n+1)!} \left| x - a \right|^{n+1}.$$

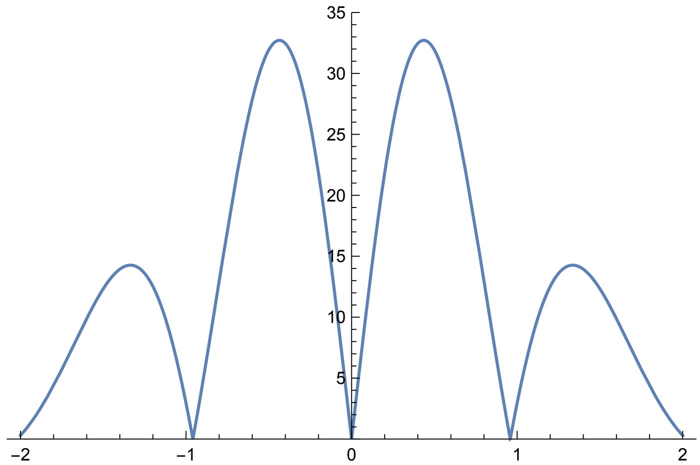
You can think of this roughly as the error on the interval is smaller than the largest "first neglected term" on the interval.

Questions

Let $f(x) = e^{-x^2}$.

- What is the Taylor series centered at zero for e^x?
- Using the Taylor series for e^x what is the Taylor series for $f(x)$?

- What is the fourth degree Taylor polynomial for $f(x)$ centered at 0?
- Using the plot $y = \left| f^{(5)}(x) \right|$ below use the Taylor series error estimate the error in approximating $f(x)$ with the 4$^{\text{th}}$ degree Taylor polynomial



Questions

Let $g(x) = \sin(x)$.

- What is the Taylor series centered at zero for $g(x)$?
- What is a simple estimate for M in the remainder for this function? ($\left| g^{(n+1)}(x) \right| \leq M$)
- What degree Taylor polynomial for $g(x)$ will approximate it with error less than 0.01 for $|x| < 1$?
- What degree Taylor polynomial for $g(x)$ will approximate it with error less than 0.01 for $|x| < 2$?
- What degree Taylor polynomial for $g(x)$ will approximate it with error less than 0.01 for $|x| < 3$?

Questions

Let $h(x) = \ln(x)$.

- What is the Taylor series centered at 1 for $h(x)$?
- What is the interval of convergence for that Taylor series?
- What is true about any estimate for M in the remainder for this function for the values of x in the interval of convergence?
- What if we use the 5$^{\text{th}}$ degree Taylor polynomial, $T_5(x)$, centered at 1 to approximate $h(x)$ for $|x - 1| \leq 0.5$? What is a good estimate for M?
- Approximate $\ln(1.5)$ with $T_5(1.5)$. What is an estimate of the error in this approximation?

Questions

Let $f(x) = e^x$.

- What is the Taylor series centered at zero for e^x?

- If we plan to use an n^{th} degree Taylor polynomial, $T_n(x)$, to approximate $f(x)$ for $-1 \le x \le 1$, what is an estimate for M?
- Find a value of n so that $T_n(x)$ approximates $f(x)$ with error less than 0.0001 for all x with $|x| \le 1$.