Taylor Polynomials

Taylor series review

If function $f(x)$ has a power series representation centered at a, then that power series must be the Taylor series centered at a,

$$f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor polynomials

Definition

The **n^{th} degree Taylor polynomial of $f(x)$ centered at a** is the partial sum of the Taylor series that goes up to and includes the n^{th} power of $(x-a)$. If the Taylor series for $f(x)$ centered at a converges to $f(x)$ for a given value x, then the n^{th} Taylor polynomial of $f(x)$ centered at a provides a polynomial approximation to $f(x)$.

Taylor remainder

If you approximate a quantity, you need some way to analyze how good the approximation is. Consider the error = | approximation – exact | . Here the exact is a given function $g(x)$ and the approximation is the n^{th} degree Taylor polynomial of $g(x)$ centered at a.

Definition

The **n^{th} Taylor remainder of $g(x)$ centered at a** is

$$R_n(x) = g(x) - \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(x-a)^k$$

In other words the error in using the n^{th} degree Taylor polynomial to approximate the function is error = $|R_n(x)|$

Analyzing error

The Taylor series error estimate: If $\left| f^{(n+1)}(x) \right| \leq M$ for all values of x of interest, then

$$\left| R_n(x) \right| \leq \frac{M}{(n+1)!} \left| x-a \right|^{n+1}.$$

You can think of this roughly as the error on the interval is smaller than the largest "first neglected term" on the interval.

Questions

Let $g(x) = \sin(x)$.

- What is the Taylor series centered at zero for $g(x)$?
- What is a simple estimate for M in the remainder for this function? ($\left|g^{(n+1)}(x)\right| \leq M$)
- What degree Taylor polynomial for $g(x)$ will approximate it with error less than 0.01 for $|x| < 1$?
- What degree Taylor polynomial for $g(x)$ will approximate it with error less than 0.01 for $|x| < 2$?
- What degree Taylor polynomial for $g(x)$ will approximate it with error less than 0.01 for $|x| < 3$?

Questions

Let $h(x) = \ln(x)$.

- What is the Taylor series centered at 1 for $h(x)$?
- What is the interval of convergence for that Taylor series?
- What is true about any estimate for M in the remainder for this function for the values of x in the interval of convergence?
- What if we use the 5^{th} degree Taylor polynomial, $T_5(x)$, centered at 1 to approximate $h(x)$ for $|x - 1| \leq 0.5$? What is a good estimate for M?
- Approximate $\ln(1.5)$ with $T_5(1.5)$. What is an estimate of the error in this approximation?

Questions

Let $f(x) = e^x$.

- What is the Taylor series centered at zero for e^x?
- If we plan to use an n^{th} degree Taylor polynomial, $T_n(x)$, to approximate $f(x)$ for $-1 \leq x \leq 1$, what is an estimate for M?
- Find a value of n so that $T_n(x)$ approximates $f(x)$ with error less than 0.0001 for all x with $|x| \leq 1$.