Section 11.9: Functions as Power Series

Review

Questions

- Write out the first four terms of each sequence.
    - $f(x) = \sum_{n=2}^{\infty} (-1)^n \frac{n(x+3)^n}{4^n}$
    - $g(x) = \sum_{n=0}^{\infty} \frac{n(2x-5)^n}{n^2+1}$
    - $h(x) = \sum_{n=3}^{\infty} \frac{(x-10)^n}{n^n}$
- Find the center of convergence and radius of convergence for the above power series.
- Find the domains of the functions defined by the above power series.

Functions as power series

Functions related to geometric series.

Questions

Consider the power series $\sum_{n=0}^{\infty} x^n$.

- For which values of x does this series converge?
- What kind of series is this?
- For the values of x when this series converges, what does it converge to?

Questions

Consider the function with power series representation from above $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

- Replace the x in this series with $-x$. In this new series for which values of x does the new series converge? What does it converge to?
- Replace the x in the original series with $\pm x^2$. What happens?

- Replace the x in the original series with $x-1$. What happens?
- Replace the x in the original series with $-(x-1)$. What happens?

Questions

- What is a power series representation for $\frac{x^3}{1-x} = x^3 \frac{1}{1-x}$?
- What is a power series representation for $\frac{x^2}{1+2x}$?

Calculus of power series

Differentiation

If $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$ has radius of convergence R, then $f'(x) = \sum_{k=1}^{\infty} c_k\, k (x-a)^{k-1}$ and it has radius of convergence R. However, you may lose convergence at one or both endpoints of the domain.

Integration

If $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$ has radius of convergence R, then $\int f(x)\, dx = \sum_{k=1}^{\infty} \frac{c_k}{k+1} (x-a)^{k+1} + C$ and it has radius of convergence R. You may gain convergence at one or both endpoints of the domain.

Questions

Let $h(x) = \frac{1}{1+x}$.

- What is a power series representation for this function?
- For which values of x does this power series converge?
- Integrate $h(x)$ and its power series.
- For which values of x does this new power series converge?
- Approximate $\ln(0.5)$ with error less than 0.0001.
- Find a power series representation for $f(x) = x \ln(1+x)$ centered at 0.

Questions

Let $g(x) = \frac{1}{1+x^2}$.

- What is a power series representation for this function?
- For which values of x does this power series converge?
- Integrate $h(x)$ and its power series.
- For which values of x does this new power series converge?

- Approximate $\tan^{-1}(0.5)$ with error less than 0.0001.
- Find a power series representation for $h(x) = \frac{\tan^{-1}(x)}{x}$ centered at 0.
- You can now integrate $\tan^{-1}(x)$ in two ways: by parts, or by term-by-term integration. Each has its advantages.

Questions

Let $k(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

- What is $k(0)$?
- What is the domain of this function?
- Write out the first 5 terms of $k(x)$, the first 5 terms of $k'(x)$ and the first 5 terms of $\int k(x)\,dx$. What well known function is this?