Review for Exam 3

Review

Taylor polynomials

The n^{th} degree Taylor polynomial for function $f(x)$ centered at a is a partial sum for the Taylor series of $f(x)$. The polynomial provides an approximation for the function.

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The error in this approximation is called the Taylor remainder $R_n(x)$, and it satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}(x-a)^{n+1}$$

where M is some number where $M \geq \left| f^{(n+1)}(x) \right|$ for all values of x of interest.

Questions

Consider $f(x) = \sin(x)$.

- What is the fourth degree Taylor polynomial for $f(x)$ centered at 0?
- We plan to use this polynomial to approximate $\sin(x)$ for $|x| \leq \pi/2$. What is the biggest $\left| f^{(5)}(x) \right|$ can be for these values of x?
- What is an error estimate for using this approximation using the Taylor remainder error estimate?
- What is an approximation for $\sin(0.6)$ using its fourth degree Taylor polynomial?
- What is an estimate of the error in this approximation?

Exam rules

- You can have one page of notes (front and back) and you are allowed to have your favorite calculator.
- You must show **full work** on all problems, especially integrals. (I.e., if you're using your calculator to evaluate integrals, you'd better be able to show me how you could have gotten your answer without it. Otherwise it's only good for checking your work....)

Format

- Questions will be similar to homework questions and weekly assignment questions.

Topics

- Section 11.1: Sequences
- Section 11.2: Series
  - Partial sums.
  - Geometric series.
  - Divergence test.
- Section 11.3: Integral test
  - Only applies to series with nonnegative terms (well, also works for series with all negative terms, of course; the point is no changes in sign).
  - Gives us the behaviors of p-series.
  - Integral test error estimate.
- Section 11.4: Comparison test
  - Compare with p-series or geometric series.
  - If the given series is less than a known series, use an appropriate error estimate from the known series.
- Section 11.5: Alternating series test
  - Has to be an alternating series for the test to apply and has to have the terms go to zero.
  - Alternating series test error estimate.
- Section 11.6: Absolute series test
  - Remove signs and either
    - Compare to known series like p-series or geometric series, or
    - Use ratio test or root test
  - Absolute convergence versus conditional convergence
    - If removal of the signs produces a convergent series, the original series with the signs converges absolutely.
    - If removal of the signs produces a divergent series, but the original series converges by the alternating series test, then that convergence is conditional.
- Section 11.8: Power series
  - Center of convergence
  - Radius of convergence—comes from applying the ratio or root test
  - Interval of convergence—always must check the endpoints with something *other than* the ratio or root test (since it's indeterminate there).
- Section 11.9: Functions as power series (using the geometric series)
  - Power series representation for $\frac{1}{1 \pm x}$ (remember that these are only **equal** for the interval of convergence).

- Power series representation for $\ln(1 \pm x)$.
- Power series representation for $\tan^{-1}(x)$.
- Differentiating and integrating power series is allowed (and encouraged!). Differentiation may rough up the interval at the endpoints, whereas integration may smooth things out.

- Section 11.10: Taylor series
  - Taylor series centered at 0 for e^x, $\sin(x)$, $\cos(x)$.
  - If you already have a power series representation for a function, it must be its Taylor series.
- Section 11.11: Taylor polynomials
  - Approximate functions with their Taylor polynomials.
  - Taylor remainder error estimate.

Studying

- Try problems you haven't worked before from the exercises in your reference textbook.
- Look through the weekly homeworks.
- Rework problems in Imath.
- Finish any worksheets.

Sample questions

1. Consider the sequence $\frac{2}{3}, -\frac{4}{9}, \frac{6}{27}, -\frac{8}{81}, \frac{10}{243}, -\frac{12}{729}, \ldots$.

1.1. Find a formula a_n for this sequence.

1.2. Does this sequence (not the summation of them) converge? If so, to what value?

2. Analyze each of the given series, if it diverges give reasons why. If it converges, either give the exact value it converges to or give the partial sum whose value approximates the series with error less than 0.001.

2.1. $\sum_{k=0}^{\infty} 2\left(\frac{4}{3}\right)^k$

2.2. $\sum_{k=0}^{\infty} e^{-k}$

2.3. $\sum_{k=1}^{\infty} \frac{k}{k+1}$

2.4. $\sum_{k=1}^{\infty} \frac{1}{k^5}$

2.5. $\sum_{k=1}^{\infty} \frac{4}{k \cdot 5^k}$

2.6. $\sum_{k=1}^{\infty} \frac{n}{n^2+1}$

3.2. $\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$

3.3. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}$

4. Consider the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n\,5^n}$.

4.1. What is its center?

4.2. Find its radius of convergence.

4.3. What is its interval of convergence? Be sure to determine convergence at the endpoints of the interval of convergence.

5. I'm *not* giving you a formula for function $f(x)$, but I will tell you its value when $x = 1$ and the value of some of its derivatives when $x = 1$: $f(1) = 2$, $f'(1) = -1$, $f''(1) = \frac{1}{2}$, $f^{(3)}(1) = -\frac{3}{8}$.

5.1. Using the above information, what is $T_3(x)$, the 3rd degree Taylor polynomial for $f(x)$ about 1?

5.2. Using your Taylor polynomial approximation for $f(x)$, what is an approximation for $f(0)$?

5.3. An extra piece of information about this function is that $\left| f^{(4)}(x) \right| \leq 3$ for all x. Use that information to estimate the error in approximating $f(0)$ with $T_3(0)$.